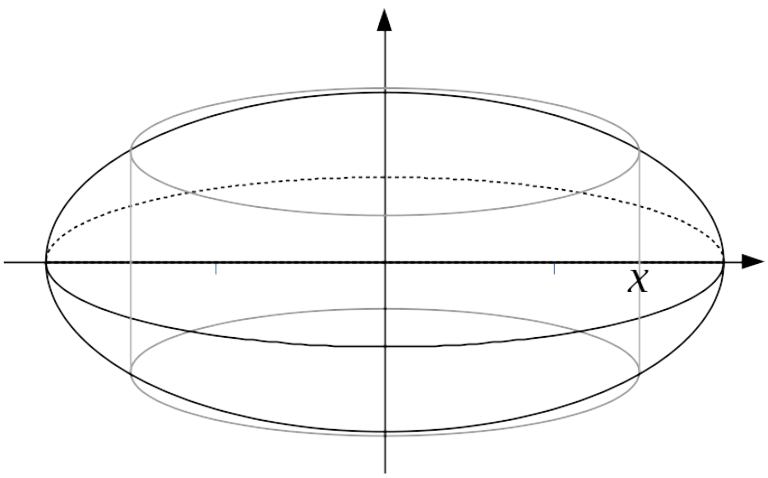
Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Summer 2020

Solutions to Quiz #4
Tuesday, 14 July.

Consider the region between $y = \sqrt{1 - \dfrac{x^2}{4}}$ and $y = -\sqrt{1 - \dfrac{x^2}{4}}$, where $0 \leq x \leq 2$. (This is the right half of the region enclosed by the ellipse $\dfrac{x^2}{4} + y^2 = 1$.) Revolve this region about the y-axis. The resulting solid of revolution is an "oblate spheroid" and looks like something like a squashed sphere.

1. Compute the volume of this oblate spheroid. *[5]*

SOLUTION. *(Cylindrical Shells)* If we use the method of cylindrical shells to compute the volume of this solid of revolution, the shells will be open cylinders with axis of symmetry the y-axis. The shells will therefore run parallel to the y-axis and perpendicular to the x-axis, so we will use x as our variable. Consider the shell that passes through x:



This shell has radius $r = x - 0 = x$ and height $h = \text{upper} - \text{lower} = \sqrt{1 - \frac{x^2}{4}} - \left(-\sqrt{1 - \frac{x^2}{4}}\right) = 2\sqrt{1 - \frac{x^2}{4}}$. From the original region, the range of x is $0 \leq x \leq 2$. It follows that the volume of the solid is given by:

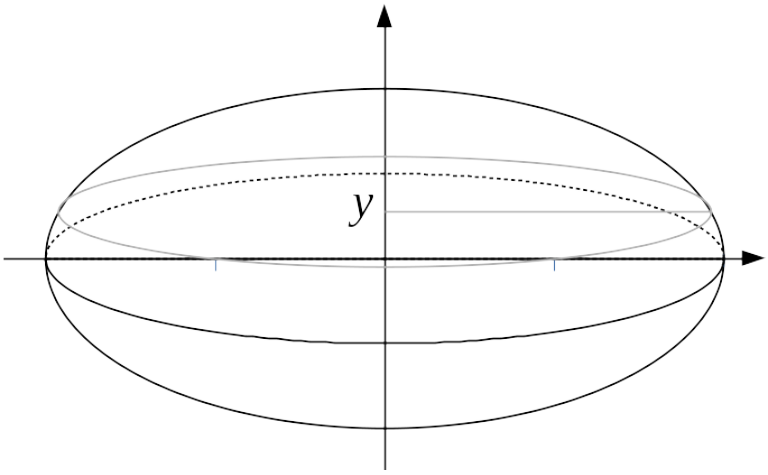
$$V = \int_0^2 2\pi r h \, dx = \int_0^2 2\pi x \cdot 2\sqrt{1 - \frac{x^2}{4}} \, dx \qquad \begin{array}{l} \text{Substitute } u = 1 - \frac{x^2}{4}, \text{ so } du = -\frac{x}{2}\,dx \\ \text{and } x\,dx = (-2)\,du, \text{ with } \begin{array}{ccc} x & 0 & 2 \\ u & 1 & 0 \end{array}. \end{array}$$

$$= \int_1^0 4\pi\sqrt{u}(-2)\,du = 8\pi \int_0^1 u^{1/2}\,du = 8\pi \cdot \left.\frac{u^{3/2}}{3/2}\right|_0^1 = \left.\frac{16}{3}\pi u^{3/2}\right|_0^1$$

$$= \frac{16}{3}\pi \cdot 1^{3/2} - \frac{16}{3}\pi \cdot 0^{3/2} = \frac{16}{3}\pi \quad \square$$

(Disks/Washers) If we use the disk/washer method to compute the volume of this solid of revolution, the disks will be centered at and stacked along the y-axis. The disks will

therefore be parallel to the x-axis and perpendicular to the x-axis, so we will use y as our basic variable. Note that in the original region, y runs from $-\sqrt{1-\frac{0^2}{4}} = -1$ to $\sqrt{1-\frac{0^2}{4}} = 1$. Consider the disk at y:



The radius of the disk at y is $r = x - 0 = x$ for the x obtained by solving $\frac{x^2}{4} + y^2 = 1$ for x in terms of y: $x^2 = 4 - 4y^2$, so $r = x = \sqrt{4-4y^2} = 2\sqrt{1-y^2}$. (We ignore the negative root, since $x \geq 0$ in the original region. Besides, a radius ought to be positive...) It follows that the volume of the solid is given by:

$$\begin{aligned} V &= \int_{-1}^{1} \pi r^2 \, dy = \int_{-1}^{1} \pi \left(2\sqrt{1-y^2}\right)^2 \, dy = \int_{-1}^{1} 4\pi \left(1-y^2\right) \, dy = 4\pi \left(y - \frac{y^3}{3}\right)\Big|_{-1}^{1} \\ &= 4\pi \left(1 - \frac{1^3}{3}\right) - 4\pi \left((-1) - \frac{(-1)^3}{3}\right) = 4\pi \cdot \frac{2}{3} - 4\pi \cdot \left(-\frac{2}{3}\right) = \frac{8}{3}\pi + \frac{8}{3}\pi = \frac{16}{3}\pi \quad \square \end{aligned}$$

2. Compute the surface area of this oblate spheroid. *[5]*

SOLUTION. Whether we use x or y as the fundamental (or "independent") variable, the surface area formula is $SA = \int_a^b 2\pi r ds$, where ds is an infinitesimal increment of arc-length. Depending on whether we choose x or y as the fundamental variable, we have $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$ or $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$. Just for fun, and because we just worked out things in terms of y in the second solution to **1** above, we will use y as the fundamental variable. In terms of y, as we noted above, $x = 2\sqrt{1-y^2}$ for $-1 \leq y \leq 1$ and x in the given region. This means that the little bit of arc at y, the ds, gets revolved around a circle of radius $r = x - 0 = x = 2\sqrt{1-y^2}$, and that

$$\begin{aligned} \frac{dx}{dy} &= \frac{d}{dy} 2\sqrt{1-y^2} = \frac{d}{dy} 2\left(1-y^2\right)^{1/2} = 2 \cdot \frac{1}{2}\left(1-y^2\right)^{-1/2} \cdot \frac{d}{dy}\left(1-y^2\right) \\ &= \left(1-y^2\right)^{-1/2} \cdot (-2y) = -2y\left(1-y^2\right)^{-1/2} \cdot = \frac{-2y}{\sqrt{1-y^2}} \end{aligned}$$

We plug all of this into the surface area formula previously mentioned:

$$
\begin{aligned}
SA &= \int_{-1}^{1} 2\pi r ds = \int_{-1}^{1} 2\pi \cdot x \cdot \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy \\
&= 2\pi \int_{-1}^{1} 2\sqrt{1-y^2} \cdot \sqrt{1+\left(\frac{-2y}{\sqrt{1-y^2}}\right)^2}\, dy \\
&= 4\pi \int_{-1}^{1} \sqrt{1-y^2} \cdot \sqrt{1+\frac{4y^2}{1-y^2}}\, dy = 4\pi \int_{-1}^{1} \sqrt{1-y^2} \cdot \sqrt{\frac{1-y^2}{1-y^2}+\frac{4y^2}{1-y^2}}\, dy \\
&= 4\pi \int_{-1}^{1} \sqrt{1-y^2} \cdot \sqrt{\frac{1+3y^2}{1-y^2}}\, dy = 4\pi \int_{-1}^{1} \sqrt{1-y^2} \cdot \frac{\sqrt{1+3y^2}}{\sqrt{1-y^2}}\, dy \\
&= 4\pi \int_{-1}^{1} \sqrt{1+3y^2}\, dy \qquad \begin{array}{l}\text{Substitute } y = \frac{1}{\sqrt{3}}\tan(t), \\ \text{so } dy = \frac{1}{\sqrt{3}}\sec^2(t)\, dt.\end{array} \\
&= 4\pi \int_{y=-1}^{y=1} \sqrt{1+3\left(\frac{1}{\sqrt{3}}\tan(t)\right)^2}\, \frac{1}{\sqrt{3}}\sec^2(t)\, dt \\
&= \frac{4\pi}{\sqrt{3}} \int_{y=-1}^{y=1} \sqrt{1+\frac{3}{3}\tan^2(t)}\, \sec^2(t)\, dt \\
&= \frac{4\pi}{\sqrt{3}} \int_{y=-1}^{y=1} \sqrt{1+\tan^2(t)}\, \sec^2(t)\, dt = \frac{4\pi}{\sqrt{3}} \int_{y=-1}^{y=1} \sqrt{\sec^2(t)}\, \sec^2(t)\, dt \\
&= \frac{4\pi}{\sqrt{3}} \int_{y=-1}^{y=1} \sec^3(t)\, dt = \frac{4\pi}{\sqrt{3}} \left[\frac{1}{2}\tan(t)\sec(t) + \frac{1}{2}\int \sec(t)\, dt\right]_{y=-1}^{y=1} \\
&= \frac{2\pi}{\sqrt{3}} \left[\tan(t)\sec(t) + \ln\left(\tan(t)+\sec(t)\right)\right]_{y=-1}^{y=1} \\
&= \frac{2\pi}{\sqrt{3}} \left[\sqrt{3}y\sqrt{1+3y^2} + \ln\left(\sqrt{3}y + \sqrt{1+3y^2}\right)\right]_{-1}^{1} \\
&= \frac{2\pi}{\sqrt{3}} \left[\sqrt{3}\cdot 1 \cdot \sqrt{1+3\cdot 1^2} + \ln\left(\sqrt{3}\cdot 1 + \sqrt{1+3\cdot 1^2}\right)\right] \\
&\quad - \frac{2\pi}{\sqrt{3}} \left[\sqrt{3}(-1)\sqrt{1+3(-1)^2} + \ln\left(\sqrt{3}(-1)y + \sqrt{1+3(-1)^2}\right)\right] \\
&= \frac{2\pi}{\sqrt{3}} \left[2\sqrt{3} + \ln\left(2+\sqrt{3}\right)\right] - \frac{2\pi}{\sqrt{3}} \left[-2\sqrt{3} + \ln\left(2-\sqrt{3}\right)\right] \\
&= \frac{2\pi}{\sqrt{3}} \left[4\sqrt{3} + \ln\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right)\right] = 8\pi + \frac{2\pi}{\sqrt{3}}\ln\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right)
\end{aligned}
$$

Not the nicest-looking of numbers ... ■