Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, Summer 2012

Solutions to Assignment #6
Solving equations with `Maple`

Recall that $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$. Its inverse function is often denoted by $\operatorname{arcsinh}(x)$.

1. Give a derivation of an expression for $\operatorname{arcsinh}(x)$ in terms of powers, roots, and the natural logarithm function. When does this expression make sense? *[5]*

HINT: This amounts to solving the equation $x = \sinh(y) = \dfrac{e^y - e^{-y}}{2}$ for y.

NOTE: You may not look up the answer for question **1**.

SOLUTION. Following the hint:

$$\begin{aligned}
x = \frac{e^y - e^{-y}}{2} &\implies 2x = e^y - e^{-y} \\
&\implies e^y - 2x - e^{-y} = 0 \\
&\implies \left(e^y\right)^2 - 2xe^y - e^{-y}e^y = 0 \\
&\implies \left(e^y\right)^2 - 2xe^y - 1 = 0 \\
&\implies e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{2x \pm \sqrt{4x^2 + 4}}{2} \\
&\implies e^y = x \pm \sqrt{x^2 + 1} \\
&\implies y = \ln\left(x \pm \sqrt{x^2 + 1}\right)
\end{aligned}$$

Note that since $\sqrt{x^2+1} \geq x$, $x - \sqrt{x^2+1} < 0$ for all x, so $\ln\left(x - \sqrt{x^2+1}\right)$ is not defined for any real x. By the same token, $x + \sqrt{x^2+1} > 0$ for all x, so $\ln\left(x + \sqrt{x^2+1}\right)$ is defined for all real x. Thus $\operatorname{arcsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right)$. ■

2. Use `Maple` to find an expression for arcsinh in terms of powers, roots, and the natural logarithm function. If it is different from the expression you obtained in answering **1**, do the two expressions really amount to the same thing or not? *[5]*

HINT: `Maple` has a command called `solve` ...

SOLUTION. Using both hints:

```
> solve(x=(exp(y)-exp(-y))/2,y)
```

$$\mathit{ln}\left(x + \sqrt{x^2+1}\right),\ \mathit{ln}\left(x - \sqrt{x^2+1}\right)$$

Note that this is the same as the solution obtained above in answer to question **1**. ■