Mathematics 1100Y – Calculus I: Calculus of one variable
Trent University, Summer 2012

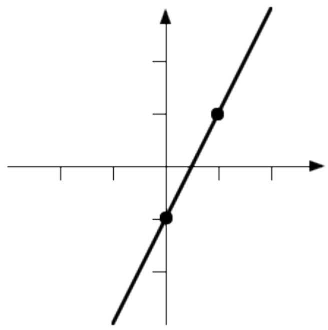
Solutions to the Quizzes

Quiz #1. Wednesday, 16 May, 2012. *[10 minutes]*

Let $f(x) = 2x - 1$.

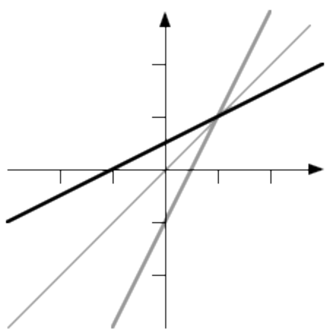
1. Sketch the graph of $f(x)$. *[2]*
2. Sketch the graph of $f^{-1}(x)$, the inverse function of $f(x)$. *[1]*
3. Find a formula for $f^{-1}(x)$. *[2]*

Solution to 1. Since $f(0) = 2 \cdot 0 - 1 = -1$ and $f(1) - 2 \cdot 1 - 1 = 1$, $(0, -1)$ and $(1, 1)$ are two points on the graph of $y = f(x)$. $f(x)$ is a linear function, so we only need to locate these two points and then draw the straight line passing through them to get the graph of $y = f(x)$:



$\blacksquare$

Solution to 2. To draw the graph of $y = f^{-1}(x)$, simply reflect the graph of $y = f(x)$ in the line $y = x$:



$\blacksquare$

Solution to 3. As usual, we try to solve $y = f(x) = 2x-1$ for x in terms of y to get an expression for $x = f^{-1}(y)$:

$$y = 2x - 1 \Longleftrightarrow 2x = y + 1 \Longleftrightarrow x = \frac{y+1}{2}$$

Thus $x = f^{-1}(y) = \dfrac{y+1}{2}$. Writing f^{-1} in terms of x as the input instead of y, we get that $f^{-1}(x) = \dfrac{x+1}{2}$. $\blacksquare$

Quiz #2. Wednesday, 23 May, 2012. *[10 minutes]*

Consider the parametric curve given by $y = \cos(2t)$ and $x = \cos(t)$, where $0 \leq t \leq \dfrac{\pi}{2}$.

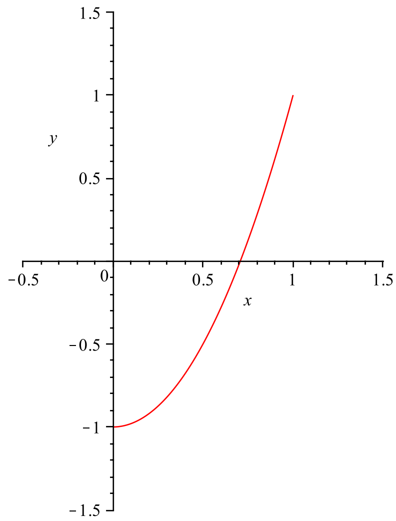
1. Show that every point on this curve is on the parabola given by $y = 2x^2 - 1$. *[3]*
2. Sketch the parametric curve. (Warning: it is not all of the parabola ...) *[2]*

Solution to 1. We'll use one of the handfull of trig identities everyone should know in this course, one of the double-angle formulas for cos:

$$y = \cos(2t) = 2\cos^2(t) - 1 = 2x^2 - 1$$

So if $y = \cos(2t)$ and $x = \cos(t)$, then $y = 2x^2 - 1$, as desired. ■

Solution to 2. Note that $\cos(t)$ runs from 1 to 0 as t runs from 0 to $\frac{\pi}{2}$; at the same time, $2t$ runs from 0 to π, so $\cos(2t)$ runs from 1 to -1. The curve thus has x values between 0 and 1 and y values between -1 and 1; it is the piece of the parabola $y = 2x^2 - 1$ which satisfies these constraints:



This graph was drawn by giving `Maple` the command:

```
plot([cos(t),cos(2*t),t=0..Pi/2],x=-0.5..1.5,y=-1.5..1.5)
```

Note that one could achieve the same end using `Maple`'s graphical user interface, but that is much harder to describe ... ■

Quiz #3. Monday, 28 May, 2012. *[10 minutes]*

1. Compute $\displaystyle\lim_{x \to 0} \frac{(x+1)\sin(x)}{x^2 + x}$. *[5]*

Solution. Here goes:

$$\lim_{x \to 0} \frac{(x+1)\sin(x)}{x^2 + x} = \lim_{x \to 0} \frac{(x+1)\sin(x)}{x(x+1)} = \lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

Note that the last step ia one of the special limits that you should take on faith for now. ■

Quiz #4. Wednesday, 30 May, 2012. *[10 minutes]*

Do *one* (1) of questions 1 or 2.

1. Compute $\displaystyle\lim_{x\to\infty} \frac{x^2+\cos(x)}{2x^2+3x}$. *[5]*
2. Let $f(x) = 3x+2$. Use the limit definition of the derivative to show that $f'(x) = 3$. *[5]*

SOLUTION TO 1. We'll divide both the numerator and denominator by the top power of x in the expression and take it from there:

$$\lim_{x\to\infty} \frac{x^2+\cos(x)}{2x^2+3x} = \lim_{x\to\infty} \frac{x^2+\cos(x)}{2x^2+3x} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{\frac{x^2}{x^2}+\frac{\cos(x)}{x^2}}{\frac{2x^2}{x^2}+\frac{3x}{x^2}} = \lim_{x\to\infty} \frac{1+\frac{\cos(x)}{x^2}}{2+\frac{3}{x}}$$
$$= \frac{\left(\lim_{x\to\infty} 1\right) + \left(\lim_{x\to\infty} \frac{\cos(x)}{x^2}\right)}{\left(\lim_{x\to\infty} 2\right) + \left(\lim_{x\to\infty} \frac{3}{x}\right)} = \frac{1+0}{2+0} = \frac{1}{2}$$

Note that $\frac{3}{x} \to 0$ as $x \to \infty$ – as x gets arbitrarily large, $\frac{3}{x}$ gets arbitrarily small – and that $\frac{\cos(x)}{x^2} \to 0$ as $x \to \infty$ by the Squeeze Theorem: since $-1 \le \cos(x) \le 1$ for all x, we have $-\frac{1}{x^2} \le \frac{\cos(x)}{x^2} \le \frac{1}{x^2}$, but $\pm\frac{1}{x^2} \to 0$ as $x \to \infty$. ■

SOLUTION TO 2. Here goes:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[3(x+h)+2]-[3x+2]}{h}$$
$$= \lim_{h\to 0} \frac{3x+3h+2-3x-2}{h} = \lim_{h\to 0} \frac{3h}{3} = \lim_{h\to 0} 3 = 3 \qquad ■$$

Quiz #5. Monday, 4 June 2012. *[10 minutes]*

1. Compute $f'(x)$ for $f(x) = \arctan\left(\dfrac{x}{x+1}\right)$. *[5]*

SOLUTION. Here goes:

$$f'(x) = \frac{d}{dx}\arctan\left(\frac{x}{x+1}\right) = \frac{1}{1+\left(\frac{x}{x+1}\right)^2} \cdot \frac{d}{dx}\left(\frac{x}{x+1}\right)$$
$$\text{(Using the Chain Rule and } \frac{d}{dt}\arctan(t) = \frac{1}{1+t^2}.\text{)}$$
$$= \frac{1}{1+\left(\frac{x}{x+1}\right)^2} \cdot \frac{\left[\frac{d}{dx}x\right]\cdot(x+1) - x\cdot\left[\frac{d}{dx}(x+1)\right]}{(x+1)^2}$$
$$\text{(Using the Quotient Rule.)}$$
$$= \frac{1}{1+\left(\frac{x}{x+1}\right)^2} \cdot \frac{1\cdot(x+1)-x\cdot 1}{(x+1)^2} = \frac{1}{1+\left(\frac{x}{x+1}\right)^2} \cdot \frac{1}{(x+1)^2}$$
$$= \frac{1}{(x+1)^2+x^2} = \frac{1}{2x^2+2x+1}$$

The last simplification is probably pointless, except as a matter of taste. ■

Quiz #6. Wednesday, 6 June, 2012. *[10 minutes]*

1. A spherical balloon is blown up, with helium being pumped into it at a constant rate of 8π m^3/s. How is the radius of the balloon changing at the moment that the radius is $\frac{1}{2}$ m? *[10]* [The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.]

SOLUTION. Let V denote the volume of the balloon. Then the given information can be summarized as $\frac{dV}{dt} = 8\pi\ m^3/s$ and $V = \frac{4}{3}\pi r^3$, and we are asked to figure out $\left.\frac{dr}{dt}\right|_{r=1/2\ m}$. To relate $\frac{dV}{dt}$ to $\frac{dr}{dt}$ we differentiate both sides of the volume formula with the help of the Chain Rule:

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi 3r^2\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$$

Plugging in what we know tells us that when $r = \frac{1}{2}$:

$$8\pi = \frac{dV}{dt} = 4\pi\left(\frac{1}{2}\right)^2\frac{dr}{dt} = 4\pi\frac{1}{4}\frac{dr}{dt} = \pi\frac{dr}{dt}$$

Solving this for $\frac{dr}{dt}$, we get that $\left.\frac{dr}{dt}\right|_{r=1/2\ m} = 8\ m/s$. ■

Quiz #7. ~~Monday, 11~~ Wednesday, 13 June, 2012. *[10 minutes]*

1. Find the maxima and minima of $g(t) = \dfrac{t^2-1}{t^2+1}$ on the interval $[-2, 1]$. *[5]*

SOLUTION. We find the critical points first:

$$\begin{aligned} h'(t) &= \frac{d}{dt}\left(\frac{t^2-1}{t^2+1}\right) = \frac{\left[\frac{d}{dt}\left(t^2-1\right)\right]\left(t^2+1\right) - \left(t^2-1\right)\left[\frac{d}{dt}\left(t^2+1\right)\right]}{\left(t^2+1\right)^2} \qquad \text{[Quotient Rule]} \\ &= \frac{2t\left(t^2+1\right) - \left(t^2-1\right)2t}{\left(t^2+1\right)^2} = \frac{2t^3+2t-2t^3+2t}{\left(t^2+1\right)^2} = \frac{4t}{\left(t^2+1\right)^2} \end{aligned}$$

It follows that

$$h'(t) \begin{smallmatrix} < \\ = \\ > \end{smallmatrix} 0 \iff 4t \begin{smallmatrix} < \\ = \\ > \end{smallmatrix} 0 \iff t \begin{smallmatrix} < \\ = \\ > \end{smallmatrix} 0\,,$$

so $t = 0$ is the only critical point; note that it does fall inside the given interval $[-2, 1]$.

Building the usual table, with some overkill for this particular problem, we get:

x	-2	$(-2, 0)$	0	$(0, 1)$	1
$h'(t)$		$-$	0	$+$	
$h(t)$	$\frac{3}{5}$	$\downarrow$	-1	$\uparrow$	0

Looking at this table, we see that $h(0) = -1$ is a local minimum, which is also the absolute minimum of $h(t)$ on the given interval, while the absolute maximum of $h(t)$ on the given interval is $h(-1) = \frac{3}{5}$. ■

Quiz #8. Monday, 20 June, 2012. *[10 minutes]*

1. Compute the average slope of $f(x) = x^3 - x$ on the interval $[-1, 2]$ and find a point c inside this interval such that $f'(c)$ is equal to the average slope of $f(x)$ on the interval. *[5]*

SOLUTION. The average slope of $f(x) = x^3 - x$ on the interval $[-1, 2]$ is

$$\frac{\text{rise}}{\text{run}} = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{\left[2^3 - 2\right] - \left[(-1)^3 - (-1)\right]}{2+1} = \frac{[8-2] - [-1+1]}{3} = \frac{6}{3} = 2\,.$$

Since $f'(x) - 3x^2 - 1$, we need to solve the equation $3x^2 - 1 = 2$ to answer the second part of the question.

$$3x^2 - 1 = 2 \iff 3x^2 = 3 \iff x^2 = 1 \iff x = \pm 1$$

Note that $+1$ is inside the interval $[-1, 2]$, so $c = 1$ is such that $f'(c)$ is equal to the average slope of $f(x)$ on $[-1, 2]$. (-1 is an endpoint, so it's debatable whether it is *inside* the interval ... :-) ■

Quiz #9. Monday, 25 June, 2012. *[10 minutes]*

1. Compute $\displaystyle\int_0^{\pi/6} \cos(3x)\, dx$. *[5]*

SOLUTION. We will use the Substitution Rule, with $u = 3x$, so that $du = 3\,dx \implies dx = \frac{1}{3}\,du$ and $\begin{array}{ccc} x & 0 & \pi/6 \\ u & 0 & \pi/2 \end{array}$.

$$\begin{aligned}\int_0^{\pi/6} \cos(3x)\, dx &= \int_0^{\pi/2} \cos(u)\,\frac{1}{3}\,du = \frac{1}{3}\int_0^{\pi/2} \cos(u)\,du = \frac{1}{3}\sin(u)\Big|_0^{\pi/2} \\ &= \frac{1}{3}\sin(\pi/2) - \frac{1}{3}\sin(0) = \frac{1}{3}\cdot 1 - \frac{1}{3}\cdot 0 = \frac{1}{3} \qquad \blacksquare\end{aligned}$$

Quiz #10. Wednesday, 27 June, 2012. *[10 minutes]*

1. Find the area between $y = x^2$ and $y = x + 2$ for $0 \le x \le 6$. *[5]*

SOLUTION. First, we determine where the two curves intersect:

$$x^2 = x + 2 \implies x^2 - x - 2 = 0 \implies x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4\cdot 1\cdot(-2)}}{2\cdot 1} = \frac{1 \pm 3}{2} = -1 \text{ or } 2$$

Only $x = 2$ is between 0 and 6; we still need to check which curve is above the other on $[0, 2]$ and $[2, 6]$, respectively. Since $1^2 = 1 < 3 = 1 + 2$, $y = x + 2$ is above $y = x^2$ on $[0, 2]$, and since $3^2 = 9 > 4 = 3 + 1$, $y = x^2$ is above $y = x + 2$ on $[2, 6]$.

It follows that the area between the curves is given by:

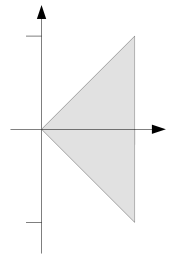
$$\begin{aligned}A &= \int_0^2 \left[(x+2) - x^2\right] dx + \int_2^6 \left[x^2 - (x+2)\right] dx = \int_0^2 \left[-x^2 + x + 2\right] dx + \int_2^6 \left[x^2 - x - 2\right] dx \\ &= \left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]\Bigg|_0^2 + \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]\Bigg|_2^6 \\ &= \left[-\frac{8}{3} + \frac{4}{2} + 4\right] - \left[-\frac{0}{3} + \frac{0}{2} - 0\right] + \left[\frac{216}{3} - \frac{36}{2} - 12\right] - \left[\frac{8}{3} - \frac{4}{2} - 4\right] = \frac{38}{3} \qquad \blacksquare\end{aligned}$$

Quiz #11. Wednesday, 4 July, 2012. *[15 minutes]*

Do *one* (1) of questions 1 or 2.

1. Sketch the region which, in polar coordinates, is between $r = 0$ and $r = \sec(\theta)$ for $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$ and find its area. *[5]*
2. Sketch the solid obtained by revolving the region between $y = 0$ and $y = \sqrt{x}$ for $0 \le x \le 4$ about the x-axis and find its volume. *[5]*

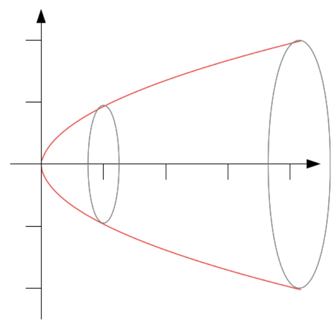
Solution to 1. Here's a sketch of the given region:



To compute the area of the region, we use the standard area formula in polar coordinates:

$$\begin{aligned}\text{Area} &= \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2\, d\theta = \int_{-\pi/4}^{\pi/4} \frac{1}{2}\sec^2(\theta)\, d\theta = \frac{1}{2}\tan(\theta)\Big|_{-\pi/4}^{\pi/4} \\ &= \frac{1}{2}\tan\left(\frac{\pi}{4}\right) - \frac{1}{2}\tan\left(-\frac{\pi}{4}\right) = \frac{1}{2}\cdot 1 - \frac{1}{2}\cdot(-1) = \frac{1}{2} + \frac{1}{2} = 1 \qquad \blacksquare\end{aligned}$$

Solution to 2. Here's a sketch of the solid:



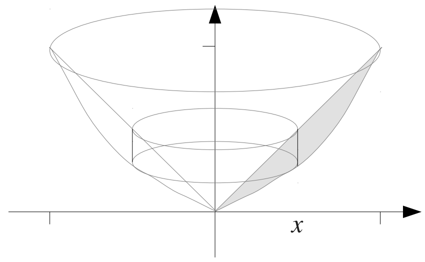
To compute the volume of the solid, we use the disk method. Note that since we are revolving the region about the x-axis, the disk method requires that we use x as the independent variable. The outer and inner radii of the disk at x is then $R = \sqrt{x} - 0 = \sqrt{x}$ and $r = 0 - 0 = 0$.

$$\begin{aligned}\text{Volume} &= \int_0^4 \pi\left(R^2 - r^2\right)\, dx = \int_0^4 \pi\left(\left[\sqrt{x}\right] - 0^2\right)\, dx = \pi\int_0^4 x\, dx = \pi\frac{x^2}{2}\Big|_0^4 \\ &= \pi\frac{4^2}{2} - \pi\frac{0^2}{2} = \frac{16}{2}\pi - 0\pi = 8\pi \qquad \blacksquare\end{aligned}$$

Quiz #12. Monday, 9 July, 2012. *[10 minutes]*

1. Sketch the solid obtained by revolving the region below $y = x$ and above $y = x^2$ for $0 \leq x \leq 1$ about the y-axis and find its volume. *[5]*

SOLUTION. Here's a sketch of the solid:



We will use the method of cylindrical shells to find the volume of the solid. Since we revolved about a vertical line, the fact that we are using the shell method means that we need to use x as the variable. The cylindrical shell at x has radius $r = x - 0 = x$ and height $h = x - x^2$. (Note that $x \geq x^2$ for $0 \leq x \leq 1$.) Plugging these into the volume formula for the shell method gives:

$$V = \int_0^1 2\pi rh\, dx = \int_0^1 2\pi x \left(x - x^2\right)\, dx = 2\pi \int_0^1 \left(x^2 - x^3\right)\, dx = 2\pi \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Bigg|_0^1$$
$$= 2\pi \left(\frac{1^3}{3} - \frac{1^4}{4}\right) - 2\pi \left(\frac{0^3}{3} - \frac{0^4}{4}\right) = 2\pi \cdot \frac{1}{12} - 2\pi \cdot 0 = \frac{\pi}{6} \qquad \blacksquare$$

Quiz #13. Wednesday, 11 July, 2012. *[12 minutes]*

1. Compute $\int \sec^4(x)\, dx$. *[5]*

SOLUTION 1. *(Trig identity and substitution)* We'll use the trigonometric identity $\sec^2(x) = 1 + \tan^2(x)$ and the substitution $w = \tan(x)$, so $dw = \sec^2(x)\, dx$.

$$\int \sec^4(x)\, dx = \int \sec^2(x) \sec^2(x)\, dx \int \left(1 + \tan^2(x)\right) \sec^2(x)\, dx = \int \left(1 + w^2\right)\, dw$$
$$= w + \frac{w^3}{3} + C = \tan(x) + \frac{1}{3}\tan^3(x) + C \qquad \square$$

SOLUTION 2. *(Integration by parts, trig identity, and algebra)* In setting up integration by parts we'll use $u = \sec^2(x)$ and $v' = \sec^2(x)$, so $u' = 2\sec(x)\frac{d}{dx}\sec(x) = 2\sec(x) \cdot \sec(x)\tan(x) = 2\sec^2(x)\tan(x)$ and $v = \tan(x)$. We'll use the trig identity $\tan^2(x) = \sec^2(x) - 1$ later on.

$$\int \sec^4(x)\, dx = \int \sec^2(x)\sec^2(x)\, dx = \int uv'\, dx = uv - \int u'v\, dx$$
$$= \sec^2(x)\tan(x) - \int 2\sec^2(x)\tan(x)\tan(x)\, dx$$
$$= \sec^2(x)\tan(x) - 2\int \sec^2(x)\tan^2(x)\, dx$$
$$= \sec^2(x)\tan(x) - 2\int \sec^2(x)\left(\sec^2(x) - 1\right)\, dx$$
$$= \sec^2(x)\tan(x) - 2\int \sec^4(x)\, dx + 2\int \sec^2(x)\, dx$$
$$= \sec^2(x)\tan(x) - 2\int \sec^4(x)\, dx + 2\tan(x)$$

Solving for the integral we're interested in, it follows that

$$3\int \sec^4(x)\,dx = \sec^2(x)\tan(x) + 2\tan(x)\,,$$

so

$$\int \sec^4(x)\,dx = \frac{1}{3}\sec^2(x)\tan(x) + \frac{2}{3}\tan(x) + C\,.$$

(The "$+C$" is a belated recognition that we're computing an indefinite integral ...) $\square$

SOLUTION 3. *(Integration by parts and substitution)* Combining the use of integration by parts from the preceding solution and the substitution from the one before that:

$$\begin{aligned}\int \sec^4(x)\,dx &= \int \sec^2(x)\sec^2(x)\,dx = \int uv'\,dx = uv - \int u'v\,dx \\ &= \sec^2(x)\tan(x) - \int 2\sec^2(x)\tan(x)\tan(x)\,dx \\ &= \sec^2(x)\tan(x) - 2\int \sec^2(x)\tan^2(x)\,dx \\ &= \sec^2(x)\tan(x) - 2\int w^2\,dw \\ &= \sec^2(x)\tan(x) - 2\cdot\frac{w^3}{3} + C \\ &= \sec^2(x)\tan(x) - \frac{2}{3}\tan^3(x) + C \qquad \square\end{aligned}$$

There are, of course, many other possible solutions, including the use of the reduction formula for $\int \sec^n(x)\,dx$. Those so inclined can amuse themselves by showing that all of these solutions are really the same ... ■

Quiz #14. Wednesday, 18 July, 2012. *[15 minutes]*

Do *one* (1) of questions 1 or 2.

1. Compute $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx$. *[5]*
2. Compute $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$. *[5]*

SOLUTION TO 1. We'll use the trigonometric substitution $x = \tan(\theta)$, so $dx = \sec^2(\theta)\,d\theta$.

$$\begin{aligned}\int \frac{1}{\sqrt{1+x^2}}\,dx &= \int \frac{1}{\sqrt{1+\tan^2(\theta)}}\,\sec^2(\theta)\,d\theta = \int \frac{1}{\sqrt{\sec^2(\theta)}}\,\sec^2(\theta)\,d\theta = \int \frac{\sec^2(\theta)}{\sec(\theta)}\,d\theta \\ &= \int \sec(\theta)\,d\theta = \ln\left(\tan(\theta)+\sec(\theta)\right) + C = \ln\left(x + \sqrt{1+x^2}\right) + C\end{aligned}$$

Note the implicit use of the calculation $\sqrt{1+x^2} = \sqrt{1+\tan^2(\theta)} = \sqrt{\sec^2(\theta)} = \sec(\theta)$ in reverse when substituting back in terms of x. ■

SOLUTION TO 2. Here goes:

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}\,dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t x^{-2}\,dx = \lim_{t\to\infty} \left.\frac{x^{-1}}{-1}\right|_1^t = \lim_{t\to\infty} \left.\frac{-1}{x}\right|_1^t \\ &= \lim_{t\to\infty}\left[\frac{-1}{t} - \left(\frac{-1}{1}\right)\right] = \lim_{t\to\infty}\left[1 - \frac{1}{t}\right] = 1 - 0 = 1\end{aligned}$$

... since $\frac{1}{t} \to 0$ as $t \to \infty$. ■

Quiz #15. Monday, 23 July, 2012. *[15 minutes]*

1. Compute $\displaystyle\int \frac{1}{x^3+x}\,dx$. *[5]*

Solution. Zeroth, the numerator, $p(x) = 1$, is a polynomial of degree 0, which is less than the degree of the denominator, $q(x) = x^3 + x$, namely 3. This means we do not have to divide the denominator into the numerator and can just dive into partial fractions.

First, we factor the denominator as far as it goes: $x^3 + x = x\left(x^2+1\right)$. Note that x is linear and x^2+1 is an irreducible quadratic. (Note that $x^2 + 1 \geq 1$ no matter what value x is given, so it has no roots, and hence is irreducible.) This means that the partial fraction decomposition has the form

$$\frac{1}{x^3+x} = \frac{1}{x\left(x^2+1\right)} = \frac{Ax+B}{x^2+1} + \frac{C}{x}\,.$$

Second, we determine the unknown coefficients A, B, and C. Putting the partial fraction decomposition over a common denominator of x^3+x and equating numerators gives us the following equation:

$$0x^2 + 0x + 1 = 1 = (Ax+B)x + C\left(x^2+1\right) = (A+C)x^2 + Bx + C$$

Since two polynomials are equal exactly when all the coefficients of corresponding powers are equal, we have that $A + C = 0$, $B = 0$, and $C = 1$, from which it follows that $A = -C = -1$. Thus

$$\frac{1}{x^3+x} = \frac{1}{x\left(x^2+1\right)} = \frac{-x}{x^2+1} + \frac{1}{x}\,.$$

Third, we compute the integral, in part with the help of the substitution $u = x^2 + 1$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$.

$$\begin{aligned}
\int \frac{1}{x^3+x}\,dx &= \int \left(\frac{-x}{x^2+1} + \frac{1}{x}\right)dx = \int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx \\
&= \ln(x) - \int \frac{1}{u}\cdot\frac{1}{2}\,du = \ln(x) - \frac{1}{2}\ln(u) + K \\
&= \ln(x) - \frac{1}{2}\ln\left(x^2+1\right) + K = \ln(x) - \ln\left(\sqrt{x^2+1}\right) + K \\
&= \ln\left(\frac{x}{\sqrt{x^2+1}}\right) + K
\end{aligned}$$

The last couple of steps are just for show ... Note the use of K instead of C for the generic constant to avoid confusion with the use of C above. ■

Quiz #16. Wednesday, 25 July, 2012. *[15 minutes]*

Do *one* (1) of questions 1 or 2.

1. Find the arc-length of the curve given in polar coordinates by $r = \theta^2$, where $0 \le \theta \le \sqrt{5}$. *[5]*
2. Find the area of the surface obtained by revolving the curve $y = \frac{2}{3}x^{3/2}$, where $0 \le x \le 1$, about the y-axis. *[5]*

Solution to 1. We plug $\frac{dr}{d\theta} = \frac{d}{d\theta}\theta^2 = 2\theta$ into the polar version of the arc-length formula. Along the way we will use the substitution $u = \theta^2 + 4$, so $du = 2\theta\, d\theta$ and hence $\theta\, d\theta = \frac{1}{2}\, du$, and $\begin{array}{ccc} \theta & 0 & \sqrt{5} \\ u & 4 & 9 \end{array}$.

$$\begin{aligned}
\text{arc-length} &= \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{\sqrt{5}} \sqrt{\left(\theta^2\right)^2 + (2\theta)^2}\, d\theta = \int_0^{\sqrt{5}} \sqrt{\theta^4 + 4\theta^2}\, d\theta \\
&= \int_0^{\sqrt{5}} \sqrt{\theta^2\left(\theta^2 + 4\right)}\, d\theta = \int_0^{\sqrt{5}} \theta\sqrt{\theta^2 + 4}\, d\theta = \int_4^9 \sqrt{u}\, \frac{1}{2}\, du = \frac{1}{2}\int_4^9 u^{1/2}\, du \\
&= \frac{1}{2} \cdot \frac{2}{3} u^{3/2}\Big|_4^9 = \frac{1}{3}9^{3/2} - \frac{1}{3}4^{3/2} = \frac{1}{3}3^3 - \frac{1}{3}2^3 = \frac{1}{3}(27 - 8) = \frac{19}{3} \qquad \blacksquare
\end{aligned}$$

Solution to 2. We plug $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2}{3}x^{3/2}\right) = \frac{2}{3} \cdot \frac{3}{2}x^{1/2} = \sqrt{x}$ into the surface area formula. Note that since we are rotating about the y-axis, we will have $r = x - 0 = x$. Along the way we will use the substitution $u = x + 1$, so $du = dx$, $x = u - 1$, and $\begin{array}{ccc} x & 0 & 1 \\ u & 1 & 2 \end{array}$.

$$\begin{aligned}
\text{area} &= \int_a^b 2\pi r\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 2\pi x\sqrt{1 + \left(\sqrt{x}\right)^2}\, dx = 2\pi\int_0^1 x\sqrt{1 + x}\, dx \\
&= 2\pi\int_1^2 (u - 1)\sqrt{u}\, du = 2\pi\int_1^2 (u - 1)u^{1/2}\, du = 2\pi\int_1^2 \left(u^{3/2} - u^{1/2}\right)\, du \\
&= 2\pi\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Big|_1^2 = 2\pi\left(\frac{2}{5}2^{5/2} - \frac{2}{3}2^{3/2}\right) - 2\pi\left(\frac{2}{5}1^{5/2} - \frac{2}{3}1^{3/2}\right) \\
&= 2\pi\left(\frac{2}{5}4\sqrt{2} - \frac{2}{3}2\sqrt{2}\right) - 2\pi\left(\frac{2}{5} - \frac{2}{3}\right) = 2\pi\frac{4}{15}\sqrt{2} - 2\pi\left(-\frac{4}{15}\right) = \frac{4}{15}\pi\left(\sqrt{2} + 1\right) \qquad \blacksquare
\end{aligned}$$

Quiz #17. Take-Home! Due on Monday, 30 July, 2012. *[5 days]*

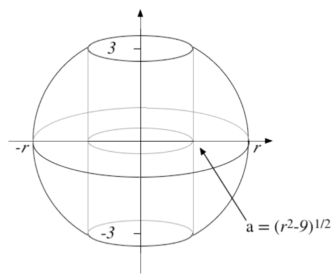
1. A cylindrical hole is drilled through a sphere, with the centre line of the cylinder passing through the centre of the sphere. After the drilling is completed, the cylindrical hole in the remaining solid is exactly 6 *cm* high. Determine the volume of the remaining solid. *[5]*

Hint: The volume of the remaining solid is $36\pi\ cm^3$.

SOLUTION. This solid can be obtained by rotating the region between the circle $x^2 + y^2 = r^2$ and the line $x = a$, with a chosen so that $a^2 + (6/2)^2 = a^2 + 9 = r^2$, about the y-axis. We can solve for the necessary a in terms of r and h:

$$a^2 + (h/2)^2 = r^2 \quad \Longrightarrow \quad a^2 = r^2 - (h/2)^2 = r^2 - h^2/4 \quad \Longrightarrow \quad a = \sqrt{r^2 - h^2/4}$$

Here's a sketch of the sucker:



We will find the volume of this solid of revolution using the washer method. Since the region was rotated about the y-axis, we need to integrate with respect to y; note that the limits for y will be -3 and 3. To avoid confusion with the r we already have in the problem, namely the radius of the sphere, we will use S for the outside radius of the washer at y and s for the inside radius. Then $S = x$ for the x such that $x^2 + y^2 = r^2$, so $S = \sqrt{r^2 - y^2}$, and $s = a = \sqrt{r^2 - 9}$. Plugging all this into the volume formula for washers gives:

$$\begin{aligned}
\text{Volume} &= \int_{-3}^{3} \pi \left[S^2 - s^2\right] dy = \pi \int_{-3}^{3} \left[\left(\sqrt{r^2 - y^2}\right)^2 - \left(\sqrt{r^2 - 9}\right)^2\right] dy \\
&= \pi \int_{-3}^{3} \left[\left(r^2 - y^2\right) - \left(r^2 - 9\right)\right] dy = \pi \int_{-3}^{3} \left[9 - y^2\right] dy \\
&\qquad \text{(It should now be apparent that the answer will } \textit{not} \text{ involve } r \ldots \text{)} \\
&= \pi \left[9y - \frac{y^3}{3}\right]\Bigg|_{-3}^{3} = \pi \left[9 \cdot 3 - \frac{3^3}{3}\right] - \pi \left[9 \cdot (-3) - \frac{(-3)^3}{3}\right] \\
&= \pi\,[27 - 9] - \pi\,[-27 + 9] = 18\pi + 18\pi = 36\pi\ cm^3 \qquad \blacksquare
\end{aligned}$$

ALTERNATE SOLUTION. If the answer does not involve r, as the hint tells us, the value of r shouldn't matter, so you can just pick one. The most convenient one is the smallest one possible, namely $3 = 6/2$. A sphere of radius $3 = 6/2$ will have height 6; in this case the cylindrical hole has to have width 0, so it takes away nothing from the volume of the sphere. Since a sphere of radius r has volume $\frac{4}{3}\pi r^3$, it follows that the sphere of radius $r = 3$ has volume $\frac{4}{3}\pi 3^3 = \frac{4}{3}\pi \cdot 27 = 36\pi$. As the value of r doesn't really matter, any solid of the sort considered in the original question should have this volume. $\square$

NOTE: This problem was adapted (very slightly) from one of Martin Gardner's columns on recreational mathematics in *Scientific American.* (Gardner, in turn, apparently got it from a periodical called *The Graham Dial,* and traced it back to a book called *Mathematical Nuts* by Samuel I. Jones.) When it appeared in Gardner's column – without a hint! – something very close to the alternate solution above was given by John W. Campbell, Jr., the editor of the science-fiction magazine *Astounding* (now called *Analog*).

Quiz #18. Monday, 30 July, 2012. *[15 minutes]*

Do *one* (1) of questions 1 or 2.

1. Compute $\displaystyle\lim_{n\to\infty} \frac{\cos(n)}{n!}$. *[5]*
2. Compute $\displaystyle\sum_{n=0}^{\infty} \pi e^{-n}$.

SOLUTION TO 1. Since $-1 \le \cos(n) \le 1$ and $n! > 0$ for all $n > 0$, we have $-\dfrac{1}{n!} \le \dfrac{\cos(n)}{n!} \le \dfrac{1}{n!}$. As $\dfrac{1}{n!} \to 0$ as $n \to \infty$ (note that $n! \ge n$), it follows by the Squeeze Theorem that $\displaystyle\lim_{n\to\infty} \frac{\cos(n)}{n!} = 0$. ■

SOLUTION TO 2. The given series is a geometric series with initial term $a = \pi e^{-0} = \pi$ and common ratio $r = e^{-1} = \frac{1}{e}$ (since $\pi e^{-(n+1)} = \pi e^{-n-1} = e^{-1}\pi e^{-n}$). Note that because $|r| = e^{-1} = \frac{1}{e} < 1$, this geometric series must converge; plugging it into the formula for the sum of a geometric series gives

$$\sum_{n=0}^{\infty} \pi e^{-n} = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} = \frac{\pi}{1-e^{-1}} = \frac{e\pi}{e-1} \qquad \blacksquare$$

Quiz #19. Wednesday, 1 August, 2012. *[15 minutes]*

Determine whether each of the following series converges or diverges.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{n+2}{n^2+3n+1}$ *[2.5]*
2. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ *[2.5]*

SOLUTION TO 1. The quickest way to do this is to use the Generalized p-Test. (Note that the terms of the series are given by rational function of n.) Since the degree of the numerator is 1 and the degree of the denominator is 2, we have $p = 2 - 1 = 1 \le 1$, so the given series diverges by the Generalized p-Test. ■

SOLUTION TO 2. We will apply the Integral Test – given what we've done so far, it's the only practical technique that does the job. In the course of computing the integral, we will use the substitution $u = \ln(x)$, so $du = \frac{1}{x}\,dx$ and $\begin{matrix} x & 1 & t \\ u & \ln(2) & \ln(t) \end{matrix}$.

$$\begin{aligned} \int_2^{\infty} \frac{1}{x\ln(x)}\,dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x\ln(x)}\,dx = \lim_{t\to\infty} \int_{\ln(2)}^{\ln(t)} \frac{1}{u}\,du = \lim_{t\to\infty} \ln(u)\Big|_{\ln(2)}^{\ln(t)} \\ &= \lim_{t\to\infty} \left[\ln(\ln(t)) - \ln(\ln(2))\right] = \infty \end{aligned}$$

because as $t \to \infty$, $\ln(t) \to \infty$, so $\ln(\ln(t)) \to \infty$. It follows by the Integral Test that the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges. ■