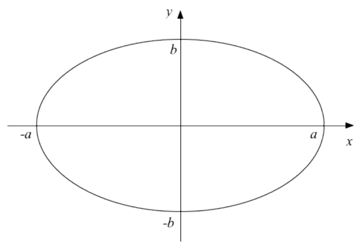
Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, Summer 2010

Solutions to Assignment #7
$\bigcirc \to \dots$

An ellipse in standard position has an equation of the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.



2. Find the area of the enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ without using calculus. *[2]*

Hint: Distort the unit circle $x^2 + y^2 = 1$ into the ellipse. How does the distortion affect areas?

SOLUTION. Consider the (linear!) transformation that sends the point (x, y) to the point $(u, v) = (ax, by)$, it i.e. stretch by a factor of a parallel to the x-axis and a factor of b parallel to the y-axis. Observe that

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1 \iff \frac{(ax)^2}{a^2} + \frac{(by)^2}{b^2} = 1 \iff x^2 + y^2 = 1\,,$$

so the transformation distorts the unit circle into the given ellipse.

How does the transformation affect areas? The unit square, with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$, which has area 1, makes an easy test case. The linear transformation takes the unit square to the rectangle with vertices $(0, 0)$, $(a, 0)$, $(0, b)$, and (a, b), which has area ab. Thus the linear transformation changes areas by a factor of ab. (You can pretty easily check, if you wish to, that this works for any square or rectangle, not just the unit square.)

The area of the unit circle is $\pi 1^2 = \pi$. It follows that the area of the given ellipse should be ab times the area of the unit circle, namely πab. ■

1. Find the area of the enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ using calculus. *[8]*

SOLUTION. We can assume that a and b are both positive. (If not, replace a by $|a|$ and/or b by $|b|$ as necessary.) Solving the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ for y, we see that the upper part of the ellipse is given by $y = b\sqrt{1 - \frac{x^2}{a^2}}$, and the lower part by $y = -b\sqrt{1 - \frac{x^2}{a^2}}$, both for $-a \le x \le a$. The area enclosed by the ellipse is then the area between these curves.

As the first step, we will use the trig substitution $x = a\sin(\theta)$, so $dx = a\cos(\theta)\,d\theta$. We will then keep the limits for x, requiring us to substitute back later. Note that it follows that $\sin(\theta) = \frac{x}{a}$, so $\cos(\theta) = \sqrt{1 - \frac{x^2}{a^2}}$ and $\theta = \arcsin\left(\frac{x}{a}\right)$.

$$\text{Area} = \int_{-a}^{a} \left[b\sqrt{1 - \frac{x^2}{a^2}} - \left(-b\sqrt{1 - \frac{x^2}{a^2}} \right) \right] dx = \int_{-a}^{a} 2b\sqrt{1 - \frac{x^2}{a^2}}\, dx$$

$$= 2b \int_{x=-a}^{x=a} \sqrt{1 - \frac{a^2 \sin^2(\theta)}{a^2}}\, a\cos(\theta)\, d\theta = 2ab \int_{x=-a}^{x=a} \sqrt{1 - \sin^2(\theta)}\, \cos(\theta)\, d\theta$$

$$= 2ab \int_{x=-a}^{x=a} \cos(\theta)\cos(\theta)\, d\theta = 2ab \int_{x=-a}^{x=a} \cos^2(\theta)\, d\theta$$

We now use the double-angle formula $\cos(2\theta) = 2\cos^2(\theta) - 1$,

so $\cos^2(\theta) = \frac{1}{2}\left(1 + \cos(2\theta)\right) \ldots$

$$= 2ab \int_{x=-a}^{x=a} \frac{1}{2}\left(1 + \cos(2\theta)\right)\, d\theta$$

$\ldots$ and the substitution $u = 2\theta$, so $du = 2d\theta$ and $d\theta = \frac{1}{2}du$.

$$= ab \int_{x=-a}^{x=a} \left(1 + \cos(u)\right) \frac{1}{2}du = \frac{1}{2}ab\, \left(u + \sin(u)\right)\Big|_{x=-a}^{x=a}$$

We now substitute back into θ, use the double-angle formula

$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \ldots$

$$= \frac{1}{2}ab\, \left(2\theta + \sin(2\theta)\right)\Big|_{x=-a}^{x=a} = \frac{1}{2}ab\, \left(2\theta + 2\sin(\theta)\cos(\theta)\right)\Big|_{x=-a}^{x=a}$$

$\ldots$ substitute back into x, and solve away.

$$= ab\left(\theta + \sin(\theta)\cos(\theta)\right)\Big|_{x=-a}^{x=a} = ab\left(\arcsin\left(\frac{x}{a}\right) + \frac{x}{a}\sqrt{1 - \frac{x^2}{a^2}} \right)\Bigg|_{x=-a}^{x=a}$$

$$= ab\left(\arcsin\left(\frac{a}{a}\right) + \frac{a}{a}\sqrt{1 - \frac{a^2}{a^2}} \right) - ab\left(\arcsin\left(\frac{-a}{a}\right) + \frac{-a}{a}\sqrt{1 - \frac{(-a)^2}{a^2}} \right)$$

$$= ab\left(\arcsin(1) + 1 \cdot \sqrt{1 - 1}\right) - ab\left(\arcsin(-1) + 1 \cdot \sqrt{1 - 1}\right)$$

$$= ab\left(\frac{\pi}{2} + 0\right) - ab\left(-\frac{\pi}{2} + 0\right) = \pi ab$$

Thus the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab. ■