Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, Summer 2010

Solution to Assignment #5
An Integral Equation

1. Assume $f(x)$ is a continuous function that satisfies the equation $(f(x))^2 = \displaystyle\int_0^x f(t)\,dt$. What function(s) could $f(x)$ be? *[10]*

Hint: Differentiate both sides and solve for $f'(x)$ first.

SOLUTION. Note first that given equation implies that $(f(0))^2 = \int_0^0 f(t)\,dt = 0$, so it must be the case that $f(0) = \pm\sqrt{0} = 0$.

Following the hint, we differentiate both sides of the given equation:

$$
\begin{aligned}
(f(x))^2 = \int_0^x f(t)\,dt \quad &\Longrightarrow \quad \frac{d}{dx}(f(x))^2 = \frac{d}{dx}\int_0^x f(t)\,dt \\
&\Longrightarrow \quad 2f(x)f'(x) = f(x) \\
&\qquad \text{[By the Chain Rule and the Fundamental Theorem.]} \\
&\Longrightarrow \quad f(x) = 0 \text{ or } 2f'(x) = 1 \\
&\qquad \text{[Since we can divide by } f(x) \text{ if it isn't 0.]} \\
&\Longrightarrow \quad f(x) = 0 \text{ or } f'(x) = \frac{1}{2} \\
&\Longrightarrow \quad f(x) = 0 \text{ or } f(x) = \frac{1}{2}x + C \\
&\qquad \text{[Since } f(x) \text{ is an antiderivative of } f'(x).] \\
&\Longrightarrow \quad f(x) = 0 \text{ or } f(x) = \frac{1}{2}x \\
&\qquad \text{[Since we must have } f(0) = 0.]
\end{aligned}
$$

Thus the only continuous functions satisfying the given integral equations are $f(x) = 0$ and $f(x) = \frac{1}{2}x$. ■