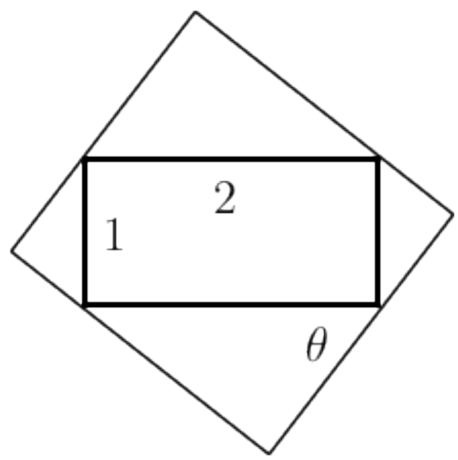
Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, Summer 2010

Solution to Assignment #2
Rectangular Surroundings

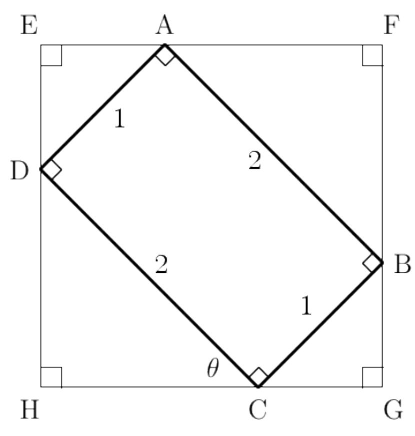
Suppose we are given a 2×1 rectangle and draw another rectangle whose sides each touch a corner of the given rectangle. (That is, the new rectangle *circumscribes* the given one.)



1. What is the maximum possible area of the new rectangle? *[10]*

Hint: Take the angle one of the sides of the new rectangle makes with one of the sides of the given rectangle, such as the angle θ in the diagram. What are the possible values of this angle? Work out the lengths of the sides of the new rectangle in terms of this angle and the dimensions of the given rectangle. Now apply max/min technology ...

SOLUTION. We will follow the hint and see where it leads. For convenience, here is a version of the picture with the corners of the rectangles duly labeled.



We'll use AB to refer to the line segment from A to B and $|AB|$ to refer to the length of this line segment.

Observe that because the sum of the interior angles of a triangle is π radians and $\angle CHD = \frac{\pi}{2}$ radians, $\angle CDH + \angle DCH = \frac{\pi}{2}$ radians. Denoting $\angle DCH = \angle HCD$ by θ, this means that $\angle CDH = \frac{\pi}{2} - \theta$.

Note too that since $\angle HCD + \angle DCB + \angle BCG = \angle HCG = \pi$ radians and $\angle DCB = \frac{\pi}{2}$ radians, $\angle HCD + \angle BCG = \frac{\pi}{2}$ radians. Thus $\angle BCG = \frac{\pi}{2} - \theta$, too.

Using the facts obtained above and applying similar reasoning we can also deduce that $\angle ADE = \angle CBG = \theta$. Thus $\cos(\theta) = \frac{|DE|}{1}$ and $\cos(\theta) = \frac{|CH|}{2}$, and $\sin(\theta) = \frac{|DH|}{2}$ and $\sin(\theta) = \frac{|CH|}{1}$. Hence

$$\begin{aligned}
|HG| &= |HC| + |CG| = 2\cos(\theta) + 1\sin(\theta) = 2\cos(\theta) + \sin(\theta) \quad \text{and} \\
|EH| &= |ED| + |DH| = 1\cos(\theta) + 2\sin(\theta) = \cos(\theta) + 2\sin(\theta)\,.
\end{aligned}$$

It follows that the area of the circumscribed rectangle is

$$\begin{aligned}
a = \text{area}(EFGH) = |EH| \cdot |HG| &= (\cos(\theta) + 2\sin(\theta))\,(2\cos(\theta) + \sin(\theta)) \\
&= 2\cos^2(\theta) + 5\cos(\theta)\sin(\theta) + 2\sin^2(\theta) \\
&= 2\left(\cos^2(\theta) + \sin^2(\theta)\right) + 5\cos(\theta)\sin(\theta) \\
&= 2 + \frac{5}{2}\sin(2\theta)\,,
\end{aligned}$$

using the trigonometric identities $\cos^2(\theta) + \sin^2(\theta) = 1$ and $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$.

We need to know what the range of possible values for θ is and where in this range critical points of the area function a occur. A look at the detailed picture should make it pretty clear that θ must be between 0 (when $DC = HG$ and the circumscribes rectangle is the given rectangle) and $\frac{\pi}{2}$ (when $DC = EH$ and the circumscribes rectangle is again the given rectangle). Note that $a = 2 + \frac{5}{2}\sin(2\theta)$ is defined and differentiable (and hence continuous) for all possible θ, so we need not consider any points except the endpoints of the interval of possible values of θ and the critical points.

Now for the critical points.

$$\frac{da}{d\theta} = \frac{d}{d\theta}\left(2 + \frac{5}{2}\sin(2\theta)\right) = 0 + \frac{5}{2}\cos(2\theta) \cdot \frac{d}{d\theta}(2\theta) = \frac{5}{2} \cdot \cos(2\theta) \cdot 2 = 5\cos(2\theta)$$

The critical points are those where $\frac{da}{d\theta} = 5\cos(2\theta) = 0$, which occurs when $2\theta = \frac{\pi}{2} \pm n\pi$, where n is an integer, *i.e.* when $\theta = \frac{\pi}{4} \pm n\frac{\pi}{2}$. The only such value in the interval $\left[0, \frac{\pi}{2}\right]$ is $\frac{\pi}{4}$, so this is the only critical point.

Comparing values,

$$\begin{aligned}
a(0) &= 2 + \frac{5}{2}\sin(2 \cdot 0) = 2 + \frac{5}{2}\sin(0) = 2 + \frac{5}{2} \cdot 0 = 2\,, \\
a\left(\frac{\pi}{4}\right) &= 2 + \frac{5}{2}\sin\left(2 \cdot \frac{\pi}{4}\right) = 2 + \frac{5}{2}\sin\left(\frac{\pi}{2}\right) = 2 + \frac{5}{2} \cdot 1 = \frac{9}{2}\,, \text{ and} \\
a\left(\frac{\pi}{2}\right) &= 2 + \frac{5}{2}\sin\left(2 \cdot \frac{\pi}{2}\right) = 2 + \frac{5}{2}\sin(\pi) = 2 + \frac{5}{2} \cdot 0 = 2\,,
\end{aligned}$$

it is clear that the maximum value of $\frac{9}{2}$ occurs when $\theta = \frac{\pi}{4}$. Thus the maximum area of a rectangle circumscribing a 2×1 rectangle is $\frac{9}{2}$. ∎