**Name:** *Solutions* **Student Number:** ________________

**Mathematics 1110H,** Trent University, Fall 2025. **The Dread Halloween Test**
a.k.a. **Test #3** for Sections F03 and F05. *Friday, 31 October.* Time: 40 minutes.

**Instructions**

- Write your name and student number at the top.
- Use only this sheet of paper, including the back side. If you need more paper, ask for it.
- You may use an aid sheet, A4- or letter-size with whatever you want written on all sides, and a calculator, with no restrictions beyond not being able to communicate with other devices.
- **Do question 1.** There is no other …

**1.** Find the domain, as well as any and all intercepts, vertical and horizontal asymptotes, intervals of increase and decrease, local maximum and minimum points, intervals of concavity, and inflection points of $f(x) = \dfrac{1}{x^2+1}$. *[5]*

Solution. *i. Domain.* Since $x^2 + 1 \geq 1 > 0$ for all $x$, the denominator is never 0. It follows that $f(x)$ is defined (and continuous and differentiable) for all $x$, *i.e.* the domain is $\mathbb{R} = (-\infty, \infty)$.

*ii. Intercepts.* $f(0) = \dfrac{1}{0^2+1} = 1$, so the $y$-intercept is 1. On the other hand, $f(x) = \dfrac{1}{x^2+1}$ is never equal to 0 because its numerator is never equal 0, so the function has no $x$-intercepts.

*iii. Vertical asymptotes.* Since $f(x)$ is defined and continuous for all $x$, it cannot have vertical asymptotes.

*iv. Horizontal asymptotes.* We take the limits as $x \to \pm\infty$ and see what happens. Note that as $x \to \pm\infty$, $x^2 + 1 \to +\infty$.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{x^2+1} \begin{matrix}\to 1\\ \to +\infty\end{matrix} = 0^+$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{1}{x^2+1} \begin{matrix}\to 1\\ \to +\infty\end{matrix} = 0^+$$

Thus $f(x)$ has $y = 0$ as a horizontal asymptote in both directions.

*v. Intervals of increase and decrease, and maxima and minima.* We first compute the derivative of $f(x)$,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{1}{x^2+1}\right) = \frac{d}{dx}\left(x^2+1\right)^{-1} = -1\left(x^2+1\right)^{-2}\frac{d}{dx}\left(x^2+1\right) \\ &= \frac{-1}{\left(x^2+1\right)^2}\cdot 2x = \frac{-2x}{\left(x^2+1\right)^2}, \end{aligned}$$

and then use it. Observe that $f'(x)$ is defined for all $x$ and that the denominator, $\left(x^2+1\right)^2$, is positive for all $x$. It follows that

$$f'(x) \begin{matrix} <0 \\ =0 \\ >0 \end{matrix} \iff -2x \begin{matrix} <0 \\ =0 \\ >0 \end{matrix} \iff \begin{matrix} x>0 \\ x=0 \\ x<0 \end{matrix},$$

so $f(x)$ is increasing for $x < 0$, has a critical point at $x = 0$ that is a local (and absolute!) maximum, and is decreasing for $x > 0$. We summarise this in the usual table:

| $x$ | $(-\infty, 0)$ | $0$ | $(0, \infty)$ |
|---|---|---|---|
| $f'(x)$ | $+$ | $0$ | $-$ |
| $f(x)$ | $\uparrow$ | max | $\downarrow$ |

Note that the maximum, $f(0) = \frac{1}{0^2+1} = 1$, is also the $y$-intercept.

*vi. Intervals of concavity and points of inflection.* We compute $f''(x)$,

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}\left(\frac{-2x}{\left(x^2+1\right)^2}\right) = \frac{\left[\frac{d}{dx}(-2x)\right]\left(x^2+1\right)^2 - (-2x)\left[\frac{d}{dx}\left(x^2+1\right)^2\right]}{\left(\left(x^2+1\right)^2\right)^2}$$

$$= \frac{-2\left(x^2+1\right)^2 + 2x \cdot 2\left(x^2+1\right) \cdot \frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^4} = \frac{-2\left(x^2+1\right)^2 + 2x \cdot 2\left(x^2+1\right) \cdot 2x}{\left(x^2+1\right)^4}$$

$$= \frac{-2\left(x^2+1\right)^2 + 8x^2\left(x^2+1\right)}{\left(x^2+1\right)^4} = \frac{-2\left(x^2+1\right) + 8x^2}{\left(x^2+1\right)^3} = \frac{6x^2-2}{\left(x^2+1\right)^3} = \frac{2\left(3x^2-1\right)}{\left(x^2+1\right)^3},$$

and then use it. Observe that $f''(x)$ is defined for all $x$ and that the denominator, $\left(x^2+1\right)^3$, is positive for all $x$. It follows that

$$f''(x) \begin{array}{l} < 0 \\ = 0 \\ > 0 \end{array} \iff 3x^2 - 1 \begin{array}{l} < 0 \\ = 0 \\ > 0 \end{array} \iff x^2 \begin{array}{l} < \frac{1}{3} \\ = \frac{1}{3} \\ > \frac{1}{3} \end{array} \iff \begin{array}{c} -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \\ x = \pm\frac{1}{\sqrt{3}} \\ x < -\frac{1}{\sqrt{3}} \text{ or } x > \frac{1}{\sqrt{3}} \end{array},$$

so $f(x)$ is concave up for $x < -\frac{1}{\sqrt{3}}$, concave down for $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$, and concave down again for $x > \frac{1}{\sqrt{3}}$, making $x = \pm\frac{1}{\sqrt{3}}$ inflection points. We'll skip the usual table this time.

*vii. Graph.* Not asked for, so we cheat by letting SageMath do the work:

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[1]: plot( 1/(x^2 + 1), -5, 5 )
[1]:
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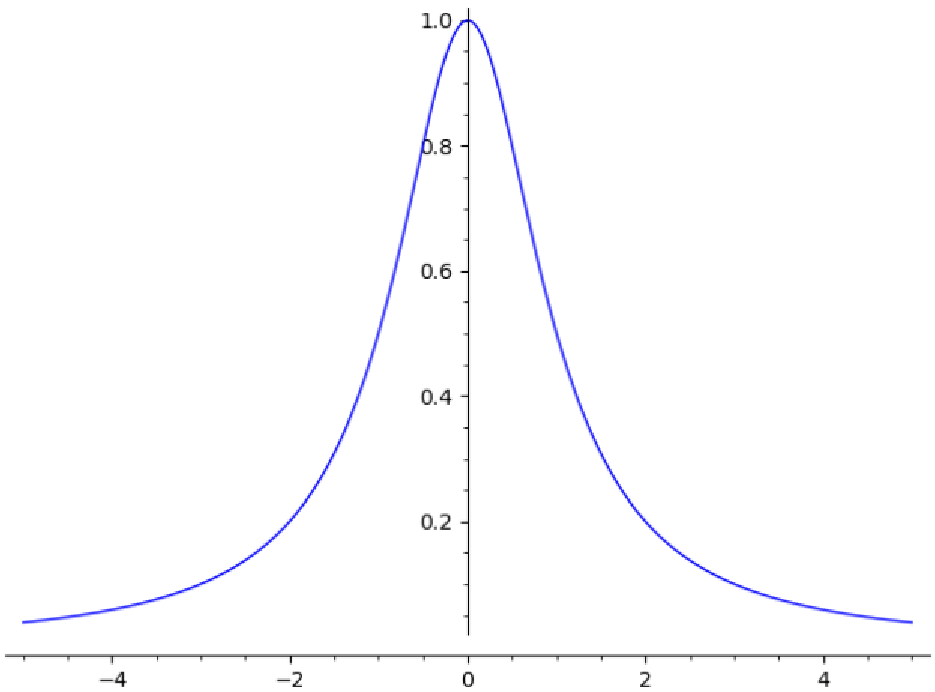


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