Name: *Solutions* **Student Number:** ______________

Mathematics 1110H, TRENT UNIVERSITY, Fall 2025. **The Dread Halloween Test**

a.k.a. **Test #3** for Section F01 and CAT. *Friday, 31 October.* Time: 40 minutes.

Instructions

- Write your name and student number at the top.
- Use only this sheet of paper, including the back side. If you need more paper, ask for it.
- You may use an aid sheet, A4- or letter-size with whatever you want written on all sides, and a calculator, with no restrictions beyond not being able to communicate with other devices.
- **Do question 1.** There is no other . . .

1. Find the domain, as well as any and all intercepts, vertical and horizontal asymptotes, intervals of increase and decrease, local maximum and minimum points, intervals of concavity, and inflection points of $f(x) = xe^x$. *[5]*

SOLUTION. *i. Domain.* $f(x) = xe^x$ is defined (and continuous and differentiable) for all $x \in \mathbb{R}$.

ii. Intercepts. $f(0) = 0e^0 = 0 \cdot 1 = 0$, so the y-intercept is 0.

On the other hand, since $e^x > 0$ for all x, $f(x) = xe^x = 0$ only when $x = 0$, so the only x-intercept is 0. Note that the origin is both the only x- and y-intercept.

iii. Vertical asymptotes. Since $f(x) = xe^x$ is defined and continuous for all x, it cannot have any vertical asymptotes.

iv. Horizontal asymptotes. We take the limits as $x \to \pm\infty$ and see what happens.

$$\begin{aligned}
&\lim_{x\to-\infty} xe^x \to -\infty \cdot 0 \\
=&\lim_{x\to-\infty} \frac{x}{e^{-x}} \begin{matrix}\to -\infty \\ \to +\infty\end{matrix} \quad \text{so we can apply l'Hôpital's Rule} \\
=&\lim_{x\to-\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}e^{-x}} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} \begin{matrix}\to 1 \\ \to -\infty\end{matrix} = 0^- \\
&\lim_{x\to+\infty} xe^x \to \infty \cdot \infty \\
=&+\infty
\end{aligned}$$

Thus $f(x) = xe^x$ has $y = 0$ as a horizaontal asymptote, but only as $x \to -\infty$.

v. Intervals of increase and decrease, and maxima and minima. We first compute $f'(x)$,

$$f'(x) = \frac{d}{dx}(xe^x) = \left[\frac{d}{dx}x\right]e^x + x\left[\frac{d}{dx}e^x\right] = 1e^x + xe^x = (1+x)e^x,$$

and then use it. Since $e^x > 0$ for all x, we have

$$f'(x) = (1+x)e^x \begin{matrix} <0 \\ =0 \\ >0 \end{matrix} \iff 1+x \begin{matrix} <0 \\ =0 \\ >0 \end{matrix} \iff \begin{matrix} x<-1 \\ x=-1 \\ x>-1 \end{matrix}.$$

It follows that $f(x) = xe^x$ is decreasing when $x < -1$ and increasing when $x > -1$, making the sole critical point at $x = -1$ a local minimum. We summmarize this in the usual table:

x	$(-\infty, -1)$	-1	$(-1, \infty)$
$f'(x)$	$-$	0	$+$
$f(x)$	$\downarrow$	min	$\uparrow$

Note that $f(-1) = -1 \cdot e^{-1} = -\frac{1}{e}$ is also the absolute minimum of $f(x)$ [Why?], and that $f(x)$ has no absolute maximum because $\lim_{x \to +\infty} xe^x = +\infty$, as noted in part *iv* above.

vi. Intervals of concavity and inflection points. We first compute $f''(x)$,

$$\begin{aligned} f''(x) &= \frac{d}{dx} f'(x) = \frac{d}{dx}\left((1+x)e^x\right) = \left[\frac{d}{dx}(1+x)\right] e^x + (1+x)\left[\frac{d}{dx}e^x\right] \\ &= 1e^x + (1+x)e^x = (2+x)e^x, \end{aligned}$$

and then use it. Since $e^x > 0$ for all x, we have

$$f''(x) = (2+x)e^x \begin{matrix} < 0 \\ = 0 \\ > 0 \end{matrix} \iff 2 + x \begin{matrix} < 0 \\ = 0 \\ > 0 \end{matrix} \iff \begin{matrix} x < -2 \\ x = -2 \\ x > -2 \end{matrix}.$$

It follows that $f(x) = xe^x$ is concave down when $x < -2$ and concave up when $x > -2$, making $x = -2$ an inflection point. We summmarize this in the usual table:

x	$(-\infty, -2)$	-2	$(-2, \infty)$
$f''(x)$	$-$	0	$+$
$f(x)$	$\frown$	infl	$\smile$

Note that $f(-2) = -2 \cdot e^{-2} = -\frac{1}{e^2}$.

vii. Graph. This wasn't actually asked for, so we cheat and let a computer draw it:

[1]:
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plot( x*e^x, -5, 2, ymin=-1, ymax=8 )
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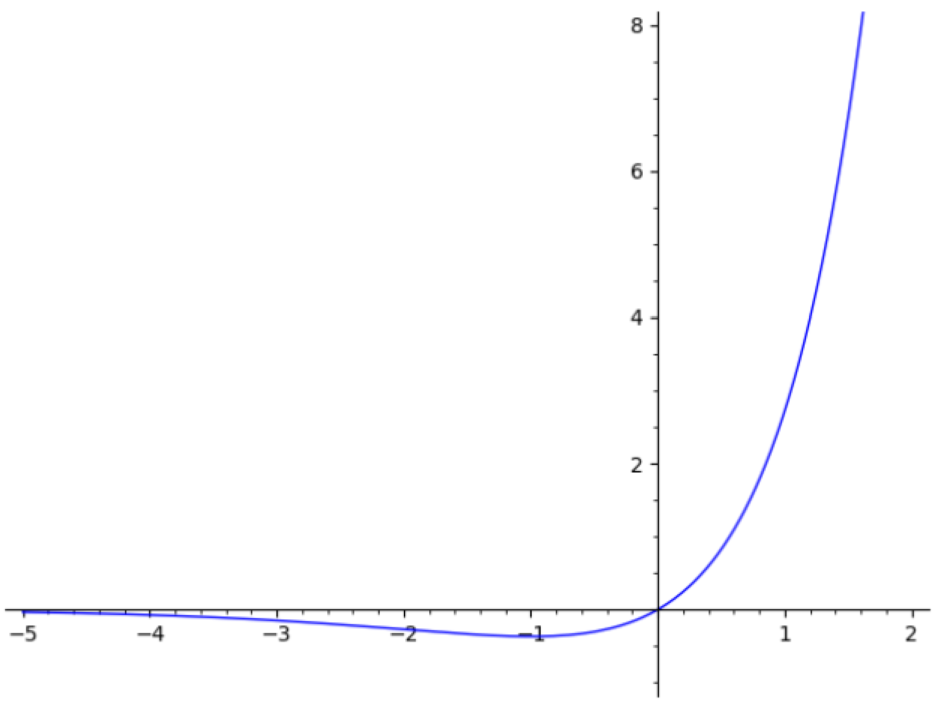
[1]:



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