Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
Section A, Trent University, Fall 2025

Solutions to the Final Examination

With some corrections.

11:00-14:00 on Monday, 8 December, in the Gym.

Instructions: Do both of parts **X** and **Y**, and, if you wish, part **Z**. Please show all your work, justify all your answers, and simplify these where you reasonably can. When you are asked to do k of n questions, only the first k that are not crossed out will be marked. *If you have a question, or are in doubt about something,* **ask!**

Aids: Any calculator, as long as it can't communicate with other devices; all sides of one letter- or A4-size sheet, with whatever you want written on it; your own brain.

Part X. Do all four (4) of **1**–**4**.

1. Compute $\frac{dy}{dx}$ as best you can in any four (4) of **a**–**f**. *[20 = 4 × 5 each]*

a. $y = \sqrt{1+x^4}$ **b.** $y = \frac{x+1}{x-1}$ **c.** $y = \left(e^x - e^{-x}\right)^2$

d. $y^2 - x^2 = 1$ **e.** $y = \ln\left(x^{41}\right)$ **f.** $y = \sec(x)\tan(x)$

Solutions. **a.** *Power Rule and Chain Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}\sqrt{1+x^4} = \frac{d}{dx}\left(1+x^4\right)^{1/2} = \frac{1}{2}\left(1+x^4\right)^{-1/2}\frac{d}{dx}\left(1+x^4\right)$$
$$= \frac{1}{2\sqrt{1+x^4}} \cdot 4x^3 = \frac{2x^3}{\sqrt{1+x^4}} \quad \square$$

b. *Quotient Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x+1}{x-1}\right) = \frac{\left[\frac{d}{dx}(x+1)\right](x-1) - (x+1)\left[\frac{d}{dx}(x-1)\right]}{(x-1)^2} = \frac{1(x-1)-(x+1)1}{(x-1)^2}$$
$$= \frac{x-1-x-1}{(x-1)^2} = \frac{-2}{(x-1)^2} \quad \square$$

c. *Power Rule and Chain Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^x - e^{-x}\right)^2 = 2\left(e^x - e^{-x}\right)\frac{d}{dx}\left(e^x - e^{-x}\right) = 2\left(e^x - e^{-x}\right)\left(\frac{d}{dx}e^x - \frac{d}{dx}e^{-x}\right)$$
$$= 2\left(e^x - e^{-x}\right)\left(e^x - e^{-x}\frac{d}{dx}(-x)\right) = 2\left(e^x - e^{-x}\right)\left(e^x - e^{-x}(-1)\right)$$
$$= 2\left(e^x - e^{-x}\right)\left(e^x + e^{-x}\right) = 2\left(\left(e^x\right)^2 + e^xe^{-x} - e^{-x}e^x - \left(e^{-x}\right)^2\right)$$
$$= 2\left(e^{2x} + e^0 - e^0 - e^{-2x}\right) = 2\left(e^{2x} - e^{-2x}\right) \quad \square$$

d. *Implicit differentiation.*

$$y^2 - x^2 = 1 \implies \frac{d}{dx}\left(y^2 - x^2\right) = \frac{d}{dx}1 \implies \frac{dy^2}{dx} - \frac{dx^2}{dx} = 0 \implies \frac{dy^2}{dy}\cdot\frac{dy}{dx} - 2x = 0$$
$$\implies 2y\frac{dy}{dx} = 2x \implies \frac{dy}{dx} = \frac{2x}{2y} = \frac{x}{y} \quad \square$$

d. *Solve for y, the Power Rule and Chain Rule.*

$$\begin{aligned} y^2 - x^2 = 1 &\implies y^2 = 1 + x^2 \\ &\implies y = \pm\sqrt{1+x^2} = \pm\left(1+x^2\right)^{1/2} \end{aligned}$$

Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\pm\left(1+x^2\right)^{1/2}\right) = \pm\frac{1}{2}\left(1+x^2\right)^{-1/2}\frac{d}{dx}\left(1+x^2\right) \\ &= \pm\frac{1}{2}\left(1+x^2\right)^{-1/2}\cdot 2x = \pm x\left(1+x^2\right)^{-1/2} = \frac{\pm x}{\sqrt{1+x^2}} \quad \square \end{aligned}$$

e. *Simplify first.*

$$\frac{dy}{dx} = \frac{d}{dx}\ln\left(x^{41}\right) = \frac{d}{dx}41\ln(x) = 41\cdot\frac{1}{x} = \frac{41}{x} \quad \square$$

e. *Chain and Power Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}\ln\left(x^{41}\right) = \frac{1}{x^{41}}\cdot\frac{d}{dx}x^{41} = \frac{1}{x^{41}}\cdot 41x^{40} = \frac{41}{x} \quad \square$$

f. *Product Rule.*

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\sec(x)\tan(x)\right) = \left[\frac{d}{dx}\sec(x)\right]\tan(x) + \sec(x)\left[\frac{d}{dx}\tan(x)\right] \\ &= \left[\sec(x)\tan(x)\right]\tan(x) + \sec(x)\left[\sec^2(x)\right] = \sec(x)\tan^2(x) + \sec^3(x) \\ &= \sec(x)\left(\sec^2(x) - 1\right) + \sec^3(x) = \sec^3(x) - \sec(x) + \sec^3(x) = 2\sec^3(x) - \sec(x) \end{aligned}$$

There are, of course, lots of ways to rewrite this answer using various trigonometric identities. ■

2. Evaluate any four (4) of the integrals **a**–**f**. *[20 = 4 × 5 each]*

a. $\displaystyle\int_0^2 (x-2)^2\,dx$ **b.** $\displaystyle\int (x\ln(x))^2\,dx$ **c.** $\displaystyle\int_0^{\pi/2} x\cos(x)\,dx$

d. $\displaystyle\int 2xe^{x^2}\,dx$ **e.** $\displaystyle\int_0^{\pi} 2\sin(x)\cos(x)\,dx$ **f.** $\displaystyle\int x\sqrt{x^2+4}\,dx$

SOLUTIONS. **a.** *Algebra and the Power Rule.* We expand the integrand first.

$$\begin{aligned}\int_0^2 (x-2)^2\,dx &= \int_0^2 \left(x^2-4x+4\right)\,dx = \left(\frac{x^3}{3}-4\frac{x^2}{2}+4x\right)\Bigg|_0^2 = \left(\frac{x^3}{3}-2x^2+4x\right)\Bigg|_0^2 \\ &= \left(\frac{2^3}{3}-2\cdot 2^2+4\cdot 2\right)-\left(\frac{0^3}{3}-2\cdot 0^2+4\cdot 0\right) \\ &= \left(\frac{8}{3}-8+8\right)-(0-0+0) = \frac{8}{3}-0=\frac{8}{3} \quad \square\end{aligned}$$

a. *Substitution and the Power Rule.* We use the substitution $w = x-2$, so $dw = dx$, and change the limits as we go: $\begin{matrix} x & 0 & 2 \\ w & -2 & 0 \end{matrix}$

$$\int_0^2 (x-2)^2\,dx = \int_{-2}^0 w^2\,dw = \frac{w^3}{3}\Bigg|_{-2}^0 = \frac{0^3}{3}-\frac{(-2)^2}{3} = 0-\frac{-8}{3}=\frac{8}{3} \quad \square$$

b. *Integration by parts, twice.* We will first use integration by parts with $u = (\ln(x))^2$ and $v' = x^2$, so $u' = 2\ln(x)\frac{1}{x}$ and $v = \frac{x^3}{3}$. The second time we use integration by parts with $s = \ln(x)$ and $t' = x^2$, so $s' = \frac{1}{x}$ and $t = \frac{x^3}{3}$.

$$\begin{aligned}\int (x\ln(x))^2\,dx &= \int x^2(\ln(x))^2\,dx = \frac{x^3}{3}(\ln(x))^2 - \int 2\ln(x)\frac{1}{x}\frac{x^3}{3}\,dx \\ &= \frac{x^3}{3}(\ln(x))^2 - \frac{2}{3}\int x^2\ln(x)\,dx \\ &= \frac{x^3}{3}(\ln(x))^2 - \frac{2}{3}\left[\frac{x^3}{3}\ln(x) - \int \frac{1}{x}\cdot\frac{x^3}{3}\,dx\right] \\ &= \frac{x^3}{3}(\ln(x))^2 - \frac{2x^3}{9}\ln(x) + \frac{2}{9}\int x^2\,dx \\ &= \frac{x^3}{3}(\ln(x))^2 - \frac{2x^3}{9}\ln(x) + \frac{2}{9}\cdot\frac{x^3}{3} + C \\ &= \frac{x^3}{3}(\ln(x))^2 - \frac{2x^3}{9}\ln(x) + \frac{x^3}{27} + C \quad \square\end{aligned}$$

c. *Integration by parts.* We will use integration by parts with $u = x$ and $v' = \cos(x)$, so $u' = 1$ and $v = \sin(x)$.

$$\begin{aligned}\int_0^{\pi/2} x\cos(x)\,dx &= x\sin(x)\big|_0^{\pi/2} - \int_0^{\pi/2} 1\sin(x)\,dx \\ &= \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) - 0\sin(0) - (-\cos(x))\big|_0^{\pi/2} = \frac{\pi}{2}1 - 0 + \cos(x)\big|_0^{\pi/2} \\ &= \frac{\pi}{2} + \cos\left(\frac{\pi}{2}\right) - \cos(0) = \frac{\pi}{2} + 0 - 1 = \frac{\pi}{2} - 1 \quad \square\end{aligned}$$

d. *Substitution.* We will us the substitution $w = x^2$, so $dw = 2x\,dx$.

$$\int 2xe^{x^2}\,dx = \int e^w\,dw = e^w + C = e^{x^2} + C \quad \square$$

e. *Substitution.* We will us the substitution $w = \sin(x)$, so $dw = \cos(x)\,dx$, changing the limits as we go: $\begin{smallmatrix} x & 0 & \pi \\ w & 0 & 0 \end{smallmatrix}$. Then

$$\int_0^{\pi} 2\sin(x)\cos(x)\,dx = \int_0^0 2w\,dw = 0$$

because the definite integral is over a single point. $\square$

e. *Substitution and the Power Rule.* We will us the substitution $w = \sin(x)$, so $dw = \cos(x)\,dx$, changing the limits as we go: $\begin{smallmatrix} x & 0 & \pi \\ w & 0 & 0 \end{smallmatrix}$. Then

$$\int_0^{\pi} 2\sin(x)\cos(x)\,dx = \int_0^0 2w\,dw = w^2\big|_0^0 = 0^2 - 0^2 = 0 \quad \square$$

e. *Trigonometric identity and substitution.* We will use the double-angle formula for sin; the substitution will be $z = 2x$, so $dz = 2\,dx$ and thus $dx = \frac{1}{2}\,dz$, and we'll change the limits as we go along: $\begin{smallmatrix} x & 0 & \pi \\ z & 0 & 2\pi \end{smallmatrix}$

$$\begin{aligned}\int_0^{\pi} 2\sin(x)\cos(x)\,dx = \int_0^{\pi} \sin(2x)\,dx = \int_0^{2\pi} \sin(z)\,\frac{1}{2}dz &= -\cos(z)\big|_0^{2\pi} \\ &= (-\cos(2\pi)) - (-\cos(0)) = (-1) - (-1) = 0 \quad \square\end{aligned}$$

f. *Substitution and the Power Rule.* We will use the substitution $w = x^2+4$, so $dw = 2x\,dx$ and thus $x\,dx = \frac{1}{2}\,dx$.

$$\begin{aligned}\int x\sqrt{x^2+4}\,dx = \int \sqrt{w}\,\frac{1}{2}\,dx = \frac{1}{2}\int w^{1/2}\,dw = \frac{1}{2}\cdot\frac{w^{3/2}}{3/2} + C &= \frac{w^{3/2}}{3} + C \\ &= \frac{1}{3}\left(x^2+4\right)^{3/2} + C \quad \blacksquare\end{aligned}$$

3. Do any four (4) of **a**–**f**. *[20 = 4 × 5 each]*

a. Find the area between $y = \sqrt{x}$ and $y = \frac{x}{2}$, where $0 \le x \le 4$.

b. Use the ε–δ definition of limits to verify that $\lim_{x \to 4} (3x - 11) = 1$.

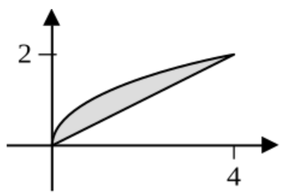
c. Compute $\displaystyle\lim_{x \to \infty} \frac{x^2}{2 + 3x^2}$.

d. Find the volume of the solid obtained by revolving the region between the line $x = 1$ and the line $y = x$, for $0 \le y \le 1$, about the x-axis.

e. Use the limit definition of the derivative to compute $\displaystyle\frac{d}{dx}(2x + 3)$.

f. Determine whether $f(x) = \begin{cases} e^{-1/x^2} & x \ne 0 \\ 0 & x = 0 \end{cases}$ is continuous at $x = 0$ or not.

Solutions. **a.** Here is a sketch of the region:



It is not hard to check that $y = \sqrt{x}$ and $y = \frac{x}{2}$ intersect at the origin and the point $(4, 2)$, and that between these points $sy = \sqrt{x}$ is above $y = \frac{x}{2}$. It follows that the area of the region is:

$$
\begin{aligned}
A &= \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right) dx = \int_0^4 \left(x^{1/2} - \frac{1}{2}x\right) dx = \left(\frac{x^{3/2}}{3/2} - \frac{1}{2} \cdot \frac{x^2}{2}\right)\Bigg|_0^4 \\
&= \left(\frac{2}{3}x^{3/2} - \frac{1}{4}x^2\right)\Bigg|_0^4 = \left(\frac{2}{3}4^{3/2} - \frac{1}{4}4^2\right) - \left(\frac{2}{3}0^{3/2} - \frac{1}{4}0^2\right) \\
&= \left(\frac{2}{3}8 - 4\right) - 0 = \frac{16}{3} - \frac{12}{3} = \frac{4}{3} \quad \square
\end{aligned}
$$

b. To verify that $\lim_{x \to 4} (3x - 11) = 1$ we need to check that for every $\varepsilon > 0$, there is a $\delta > 0$, such that if $|x - 4| < \delta$, then $|(3x - 11) - 1| < \varepsilon$. As usual, we attempt to reverse-engineer the necessary delta from the desired conclusion. Suppose an $\varepsilon > 0$ is given.

$$
\begin{aligned}
|(3x - 11) - 1| < \varepsilon &\iff |3x - 12| < \varepsilon \\
&\iff |3(x - 4)| < \varepsilon \\
&\iff 3|x - 4| < \varepsilon \\
&\iff |x - 4| < \frac{\varepsilon}{3}
\end{aligned}
$$

Now set $\delta = \frac{\varepsilon}{3}$. Then, when we have $|x-4| < \delta$, we are guaranteed that $|(3x - 11) - 1| < \varepsilon$ because every step above is fully reversible. $\square$

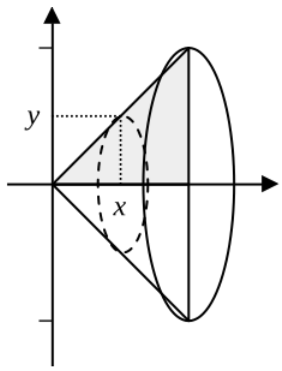
c. *Algebraically.* Here we go:

$$\lim_{x\to\infty} \frac{x^2}{2+3x^2} \begin{matrix}\to \infty \\ \to \infty\end{matrix} = \lim_{x\to\infty} \frac{x^2}{2+3x^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{\frac{2}{x^2}+3} \begin{matrix}\to 1 \\ \to 0+3\end{matrix} = \frac{1}{3} \quad \square$$

c. *Using l'Hôpital's Rule.* Here we go:

$$\lim_{x\to\infty} \frac{x^2}{2+3x^2} \begin{matrix}\to \infty \\ \to \infty\end{matrix} = \lim_{x\to\infty} \frac{\frac{d}{dx}x^2}{\frac{d}{dx}\left(2+3x^2\right)} = \lim_{x\to\infty} \frac{2x}{0+6x} = \lim_{x\to\infty} \frac{1}{3} = \frac{1}{3} \quad \square$$

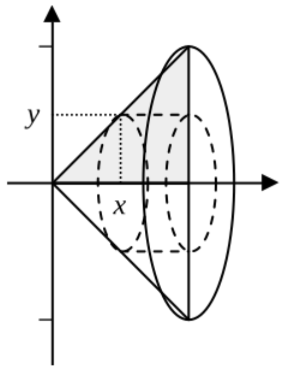
d. *Using the disk/washer method.* Here is a sketch of the solid, with a generic disk cross-section drawn in.



Since we are using disks and revolved about the x-axis, we ought to use x as the variable because the disks are perpendicular to the x-axis. Note that $0 \leq x \leq 1$ over the region we started with, and that the radius of the disk at x is $r = y - 0 = y = x$. It follows that the volume of the solid is:

$$V = \int_0^1 \pi r^2\, dx = \int_0^1 \pi x^2\, dx = \left.\frac{\pi x^3}{3}\right|_0^1 = \frac{\pi 1^3}{3} - \frac{\pi 0^3}{3} = \frac{\pi}{3} - 0 = \frac{\pi}{3} \quad \square$$

d. *Using the cylindrical shell method.* Here is a sketch of the solid, with a generic cylindrical shell drawn in.



Since we are using shells and revolved about the x-axis, we ought to use y as a variable because the shells are perpendicular to the y-axis. Note that $0 \leq y \leq 1$ over the given region, and that the shell at y has radius $r = y - 0 = y$ and height $h = 1 - x = 1 - y$. It follows that the volume of the solid is:

$$V = \int_0^1 2\pi rh \, dy = 2\pi \int_0^1 y(1-y)\, dy = 2\pi \int_0^1 \left(y - y^2\right) dy = 2\pi \left(\frac{y^2}{2} - \frac{y^3}{3} \right) \Big|_0^1$$
$$= 2\pi \left(\frac{1^2}{2} - \frac{1^3}{3} \right) - 2\pi \left(\frac{0^2}{2} - \frac{0^3}{3} \right) = 2\pi \left(\frac{1}{2} - \frac{1}{3} \right) - 2\pi \cdot 0 = 2\pi \frac{1}{6} - 0 = \frac{\pi}{3} \quad \square$$

d. *Geometry!* We worked out in class that the volume of a right circular cone with radius r and height h is $V = \frac{\pi r^2 h}{3}$. The solid of revolution in this problem is easily seen to be a cone with $r = h = 1$, so it must have volume $V = \frac{\pi 1^2 1}{3} = \frac{\pi}{3}$. $\square$

e. The limit definition of the derivative is $\frac{d}{dx} f(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. In the present instance, $f(x) = 2x + 3$, so its derivative is:

$$\frac{d}{dx}(2x+3) = \lim_{h \to 0} \frac{[2(x+h)+3] - [2x+3]}{h} = \lim_{h \to 0} \frac{2x + 2h + 3 - 2x - 3}{h}$$
$$= \lim_{h \to 0} \frac{2h}{h} = \lim_{h \to 0} 2 = 2 \quad \square$$

f. A function $f(x)$ is continuous at $x = 0$ exactly when $\lim_{x \to 0} f(x) = f(0)$. We take the limit and see what happens when $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$.

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} e^{-1/x^2} = \lim_{z \to -\infty} e^z \quad \left(\text{Since } -\frac{1}{x^2} \to -\infty \text{ as } x \to 0.\right)$$
$$= 0 = f(0)$$

It follows, by the definition of continuity at a point, that $f(x)$ is continuous at $x = 0$. ■

4. Find the domain, intercepts, vertical and horizontal asymptotes, intervals of increase and decrease, maximum and minimum points, intervals of concavity, and inflection points of $f(x) = \dfrac{x^2}{1+x^2}$, and sketch its graph based on this information. *[14]*

SOLUTION. We run through the indicated checklist.

i. Domain. Since $1 + x^2 \geq 1 > 0$ for all x, $f(x) = \dfrac{x^2}{1+x^2}$ is defined for all x. (It is also continuous and differentiable for all x.)

ii. Intercepts. $f(0) = \dfrac{0^2}{1+0^2} = \dfrac{0}{1} = 0$, so the y-intercept is $y = 0$.

$f(x) = \dfrac{x^2}{1+x^2} = 0$ only when $x^2 = 0$, which only occurs when $x = 0$. Thus the only x-intercept is also the y-intercept.

iii. Vertical asymptotes. Since $f(x)$ is defined and continuous for all x, and as vertical asymptotes are discontinuities, $f(x)$ does have any vertical asymptotes.

iv. Horizontal asymptotes. We take the limits as $x \to \pm\infty$ and see what happens.

$$\lim_{x\to-\infty} \frac{x^2}{1+x^2} = \lim_{x\to-\infty} \frac{x^2}{1+x^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to-\infty} \frac{1}{\frac{1}{x^2}+1} = \frac{1}{0^+ + 1} = 1^-$$

$$\lim_{x\to+\infty} \frac{x^2}{1+x^2} = \lim_{x\to+\infty} \frac{x^2}{1+x^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to+\infty} \frac{1}{\frac{1}{x^2}+1} = \frac{1}{0^+ + 1} = 1^-$$

Thus $f(x)$ has $x = 1$ as a horizontal asymptote in both directions, which it approaches from below in both directions.

v. Increase/decrease & maxima/minima. We first compute the derivative of $f(x)$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{x^2}{1+x^2}\right) = \frac{\left[\frac{d}{dx}x^2\right](1+x^2) - x^2\left[\frac{d}{dx}(1+x^2)\right]}{(1+x^2)^2} \\ &= \frac{[2x](1+x^2) - x^2[2x]}{(1+x^2)^2} = \frac{2x + 2x^3 - 2x^3}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2} \end{aligned}$$

Since the denominator of $f'(x)$ is defined for all x, the only critical points of $f(x)$ will be those for which $f'(x) = \dfrac{2x}{(1+x^2)^2} = 0$, which can happen only when $x = 0$. Similarly, since the denominator is always positive, $f'(x)$ is positive or negative exactly when the numerator, $2x$, is. It follows that $f'(x) < 0$, and so $f(x)$ is decreasing, exactly when $x < 0$, and $f'(x) > 0$, and so $f(x)$ is increasing, exactly when $x > 0$. Thus the critical point at $x = 0$ is a local (and absolute – why?) minimum. We summarize all this in a table:

x	$(-\infty, 0)$	0	$(0, \infty)$
$f'(x)$	$-$	0	$+$
$f(x)$	$\downarrow$	min	$\uparrow$

vi. Concavity & inflection. We first compute the second derivative of $f(x)$.

$$\begin{aligned} f''(x) &= \frac{d}{dx} f'(x) = \frac{d}{dx}\left(\frac{2x}{(1+x^2)^2}\right) = \frac{\left[\frac{d}{dx}2x\right]\left(1+x^2\right)^2 - 2x\left[\frac{d}{dx}\left(1+x^2\right)^2\right]}{\left((1+x^2)^2\right)^2} \\ &= \frac{[2]\left(1+x^2\right)^2 - 2x\left[2\left(1+x^2\right)\frac{d}{dx}\left(1+x^2\right)\right]}{(1+x^2)^4} = \frac{2\left(1+x^2\right)^2 - 2x\left[2\left(1+x^2\right)2x\right]}{(1+x^2)^4} \\ &= \frac{2\left(1+x^2\right)^2 - 8x^2\left(1+x^2\right)}{(1+x^2)^4} = \frac{2\left(1+x^2\right) - 8x^2}{(1+x^2)^3} = \frac{2-6x^2}{(1+x^2)^3} = \frac{2\left(1-3x^2\right)}{(1+x^2)^3} \end{aligned}$$

Since $\frac{2}{(1+x^2)^3}$ is defined and positive for all x, $f''(x)$ is positive, negative, or zero exactly as $1-3x^2$ is. $1-3x^2 = 0 \iff x^2 = \frac{1}{3} \iff x = \pm\frac{1}{\sqrt{3}} \approx 0.5774$. Similarly, $1-3x^2 > 0 \iff x^2 < \frac{1}{3} \iff -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$ and $1-3x^2 < 0 \iff x^2 > \frac{1}{3} \iff x < -\frac{1}{\sqrt{3}}$ or $x > \frac{1}{\sqrt{3}}$. It follows that $f(x)$ is concave up when $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$ and concave down when $x < -\frac{1}{\sqrt{3}}$ or $x > \frac{1}{\sqrt{3}}$, so both $x = -\frac{1}{\sqrt{3}}$ and $x = \frac{1}{\sqrt{3}}$ are inflection points. We summarize all this in a table:

x	$\left(-\infty, -\frac{1}{\sqrt{3}}\right)$	$-\frac{1}{\sqrt{3}}$	$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$\frac{1}{\sqrt{3}}$	$\left(\frac{1}{\sqrt{3}}, \infty\right)$
$f''(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\frown$	infl	$\smile$	infl	$\frown$

vii. The Graph. Cheating ever so slightly, we have SageMath draw the graph:

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[1]: plot( x^2/(1+x^2), -5, 5 )
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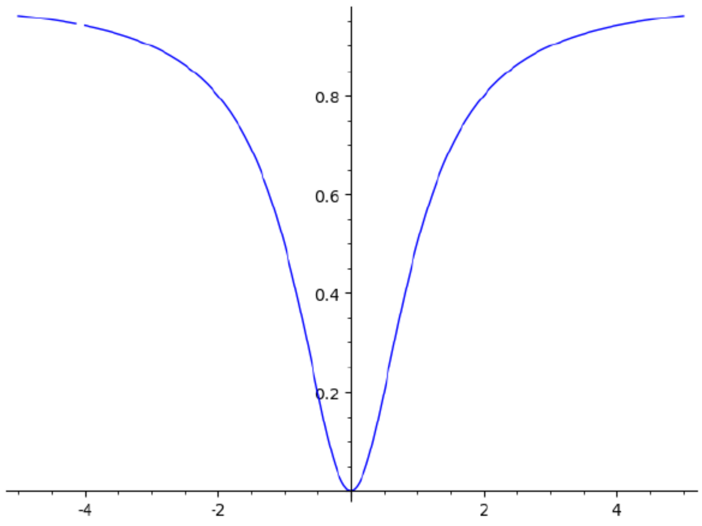
[1]:



■

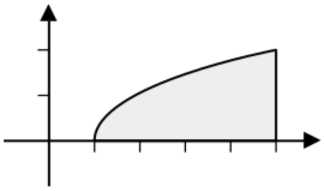
Part Y. Do any two (2) of **5–7**. *[26 = 2 × 13 each]* *Here is the "more"!*

5. The region below $y = \sqrt{x-1}$ and above $y = 0$, where $1 \le x \le 5$, is revolved about the y-axis, making a solid of revolution.

a. Sketch the region. *[1]* **b.** Find the area of the region. *[3]*

c. Sketch the solid. *[1]* **d.** Find the volume of the solid. *[8]*

SOLUTIONS. **a.** Here is a sketch of the region.

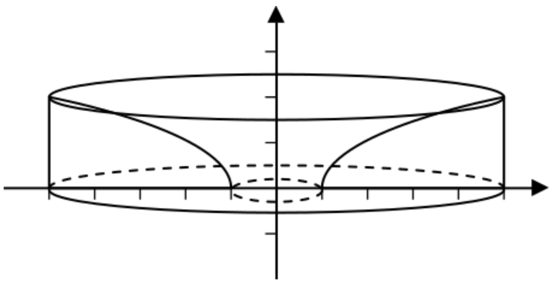


$\square$

b. We compute the area between the upper boundary and the lower boundary of the region. We will use the substitution $w = x - 1$, so $dw = dx$, and change the limits as we go: $\begin{smallmatrix} x & 1 & 5 \\ w & 0 & 4 \end{smallmatrix}$

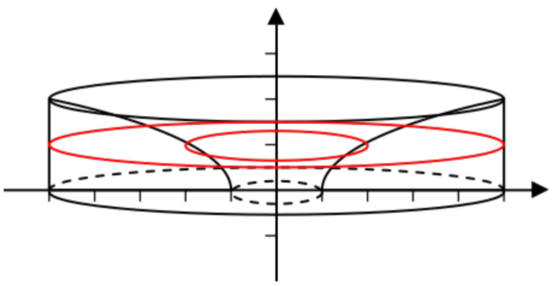
$$\text{Area} = \int_1^5 \left(\sqrt{x-1} - 0\right)\,dx = \int_1^5 (x-1)^{1/2}\,dx = \int_0^4 w^{1/2}\,dw$$
$$= \left.\frac{w^{3/2}}{3/2}\right|_0^4 = \left.\frac{2}{3}w^{3/2}\right|_0^4 = \frac{2}{3}\cdot 4^{3/2} - \frac{2}{3}\cdot 0^{3/2} = \frac{2}{3}\cdot 8 - 0 = \frac{16}{3} \quad \square$$

c. Here is a sketch of the solid:



$\square$

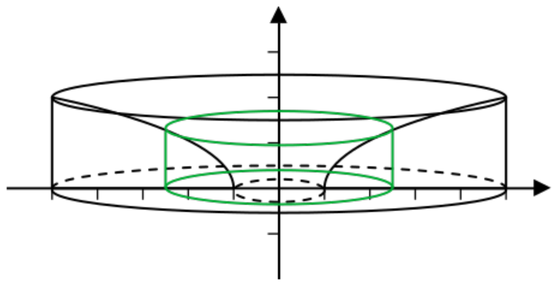
d. *Disk/washer method.* Here is a sketch of the solid with a generic washer cross-section drawn in:



Since we are using washers and revolved the region about the y-axis, we ought to use y as the variable because the washers are perpendicular to the y-axis. Note that $0 \leq y \leq 2$ over the original region, and that the outer radius of the washer at y is always $R = 5-0 = 5$, while the inner radius of the same washer is $r = x-0 = x$, where $y = \sqrt{x-1}$, so $x = y^2+1$, and thus $r = x = y^2+1$. It follows that the volume of the region is given by:

$$\begin{aligned}
V &= \int_0^2 \pi\left(R^2 - r^2\right)\,dy = \pi \int_0^2 \left(5^2 - \left(y^2+1\right)^2\right)\,dy \\
&= \pi \int_0^2 \left(25 - \left(y^4 + 2y^2 + 1\right)\right)\,dy = \pi \int_0^2 \left(-y^4 - 2y^2 + 24\right)\,dy \\
&= \pi\left(-\frac{y^5}{5} - \frac{2y^3}{3} + 24y\right)\Big|_0^2 = \pi\left(-\frac{2^5}{5} - \frac{2\cdot 2^3}{3} + 24\cdot 2\right) - \pi\left(-\frac{0^5}{5} - \frac{2\cdot 0^3}{3} + 24\cdot 0\right) \\
&= \pi\left(-\frac{32}{5} - \frac{16}{3} + 48\right) - \pi\cdot 0 = \pi\left(-\frac{96}{15} - \frac{80}{15} + \frac{720}{15}\right) - 0 \\
&= \frac{544\pi}{15} \approx 113.9351 \quad \square
\end{aligned}$$

d. *Cylindrical shell method.* Here is a sketch of the solid with a generic cylindrical shell cross-section drawn in:



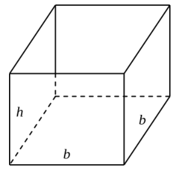
Since we are using shells and revolved the region about the y-axis, the shells are parallel to the y-axis and perpendicular to the x-axis, so we ought to use x as our variable. Note that $1 \leq x \leq 5$ over the original region, and that the shell at x has radius $r = x - 0 = x$ and height $h = y - 0 = y = \sqrt{x-1}$. It follows that the volume of the region is given by:

$$V = \int_1^5 2\pi r h\,dx = 2\pi \int_1^5 x\sqrt{x-1}\,dx$$

Substitute $w = x - 1$, so $dw = dx$, and change the limits as we go: $\begin{matrix} x & 1 & 5 \\ w & 0 & 4 \end{matrix}$

$$\begin{aligned}
&= 2\pi \int_0^4 (w+1)\sqrt{w}\,dw = 2\pi \int_0^4 \left(w^{3/2} + w^{1/2}\right)\,dw = 2\pi\left(\frac{w^{5/2}}{5/2} + \frac{w^{3/2}}{3/2}\right)\Big|_0^4 \\
&= 2\pi\left(\frac{2}{5}w^{5/2} + \frac{2}{3}w^{3/2}\right)\Big|_0^4 = 2\pi\left(\frac{2}{5}\cdot 4^{5/2} + \frac{2}{3}\cdot 4^{3/2}\right) - 2\pi\left(\frac{2}{5}\cdot 0^{5/2} + \frac{2}{3}\cdot 0^{3/2}\right) \\
&= 2\pi\left(\frac{2}{5}\cdot 32 + \frac{2}{3}\cdot 8\right) - 2\pi\cdot 0 = 2\pi\left(\frac{64}{5} + \frac{16}{3}\right) - 0 = 2\pi\left(\frac{192}{15} + \frac{80}{15}\right) \\
&= 2\pi\cdot\frac{272}{15} = \frac{544\pi}{15} \approx 113.9351 \quad \blacksquare
\end{aligned}$$

6. A small cardboard box has a square bottom and no top. If 48 cm^2 of cardboard are used to make the box, what is its maximum possible volume? What are the dimensions of such a box of maximum volume? *[13]*



SOLUTION. Suppose the box has side length b at the square base and height h. The volume of the box would then be $V = b^2h$ and its surface area would be $A = b^2 + 4bh$ (the area of the base plus the areas of the four side panels). If we use all the cardboard to make the box – which we ought to in order to get the maximum possible volume – then $A = b^2 + 4bh = 48$, which we can use to solve for h in terms of b: $h = \dfrac{48 - b^2}{4b} = \dfrac{12}{b} - \dfrac{b}{4}$. This, in turn, lets us express the volume as a function of b:

$$V = b^2h = b^2\left(\frac{12}{b} - \frac{b}{4}\right) = 12b - \frac{b^3}{4}$$

Note that we must have $0 < b \leq \sqrt{48} = 4\sqrt{3}$; on the one hand we can't have $b = 0$, since otherwise we would have $A = 0^2 + 4 \cdot 0 \cdot h = 0 \neq 48$, and on the other hand $b^2 = 48$ when $h = 0$.

The given problem therefore comes down to maximizing $V = 12b - \dfrac{b^3}{4}$ for $0 < b \leq 4\sqrt{3}$. At the endpoints we have:

$$\lim_{b \to 0^+} V(b) = \lim_{b \to 0^+}\left(12b - \frac{b^3}{4}\right) = 12 \cdot 0 - \frac{0^3}{4} = 0$$

$$V\left(4\sqrt{3}\right) = 12 \cdot 4\sqrt{3} - \frac{\left(4\sqrt{3}\right)^3}{4} = 48\sqrt{3} - \frac{192\sqrt{3}}{4} = 48\sqrt{3} - 48\sqrt{3} = 0$$

It remains to check any critical points in the interval.

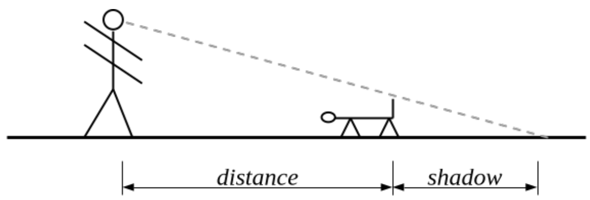
$$V'(b) = \frac{d}{db}\left(12b - \frac{b^3}{4}\right) = 12 - \frac{3}{4}b^2 = 0 \iff 48 - 3b^2 = 0 \iff b^2 = \frac{48}{3} = 16$$
$$\iff b = \pm 4$$

The critical point $b = -4 < 0$, so it is not in the interval, but $b = 4$ is in the interval since $0 < 4 < 4\sqrt{3}$. At $b = 4$, we have:

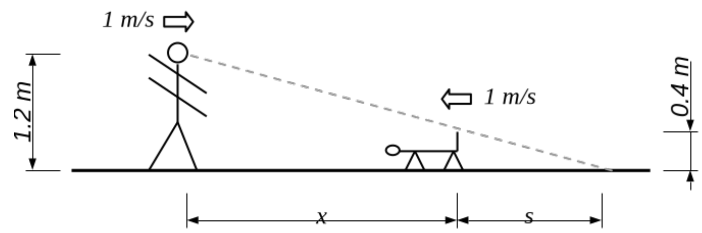
$$V(4) = 12 \cdot 4 - \frac{4^3}{4} = 48 - 16 = 32$$

Since this volume is greater than what we get at the endpoints and the volume function $V = 12b - \dfrac{b^3}{4}$ is defined and continuous for $0 < b \leq 4\sqrt{3}$, it follows that the maximum of a box meeting the goven requirements is 32 cm^3. (Recall that the area of the cardboard used to make the box was givem in cm^2.) ■

7. It is night in a dark and narrow alley. A four-armed robot, bearing a headlight 1.2 m above the pavement, moves along the alley at 1 m/s from one end, and a kitten, holding the tip of its straight-up tail 0.4 m above the pavement, moves along the alley at 1 m/s from its other end. How is the length of the shadow cast by the kitten's rear and tail changing at the instant that the robot and the kitten are 4 m apart? *[13]*



Solution. Here is an augmented version of the given diagram, with the distance between the robot and the kitten (really the kitten's tail) labeled as x and the length of the shadow labelled as s:



We are asked to work out how the length of the shadow is changing when the robot and the kitten are 4 m apart, *i.e.* $\left.\frac{ds}{dt}\right|_{x=4}$.

Looking at the diagram, it is not hard to see that we have two similar triangles: one with height 1.2 m and base $x+s$ and a smaller one with height 0.4 m and base s. (They are similar because they have a common angle at the tip of the shadow and each has a right angle at the other end of their base.) Since corresponding sides in similar triangle must have the same proportions, it follows that $\frac{x+s}{1.2} = \frac{s}{0.4}$. It follows, in turn, that

$$\frac{x+s}{1.2} = \frac{s}{0.4} \implies x+s = 1.2 \cdot \frac{x+s}{1.2} = 1.2 \cdot \frac{s}{0.4} = 3s \implies x = 3s - s = 2s \implies s = \frac{x}{2}.$$

Note that because each of the robot and the kitten is moving towards the other at constant rates of 1 m/s, the combined rate of closure is a constant 2 m/s, so $\frac{dx}{dt} = -1-1 = -2$ at every instant, including when $x = 4$. Thus, at every instant,

$$\frac{ds}{dt} = \frac{d}{dt}\left(\frac{x}{2}\right) = \frac{1}{2} \cdot \frac{dx}{dt} = \frac{1}{2} \cdot (-2) = -1,$$

that is, the length of the shadow is changing at a rate of -1 ms at every instant, including when the robot and the kitten are 4 m apart. ■

[Total = 100]

Part Z. *Bonus points!* Do one or both of **8** and **9**.

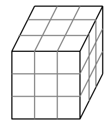
8. Write an original haiku touching on calculus or mathematics in general. *[1]*

What is a haiku?

seventeen in three:
five and seven and five of
syllables in lines

SOLUTION. For reasons which ought to be obvious, you're on your own! :-) ■

9. A dangerously sharp tool is used to cut a cube with a side length of $3\,cm$ into 27 smaller cubes with a side length of $1\,cm$. This can be done easily with six cuts. Can it be done with fewer? (Rearranging the pieces between cuts is allowed.) If so, explain how; if not, explain why not. *[1]*



~~SOLUTION~~ HINT. There are 27 smaller cubes – consider the one in the center. ■

APOLOGIES FOR ALL THE GLITCHES.
HAVE A GOOD BREAK!