Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Fall 2025

Assignment #5
Sums and Areas
Due on Friday, 21 November.[*]

Before tackling this assignment, please read the handout *Right-Hand Rule Riemann Sums*, and perhaps give the handout *A Precise Definition of the Definite Integral* a quick look as well. Section 7.1 and the first part of Section 7.2 in the textbook cover somewhat similar territory without explicitly giving the Right-Hand Rule or a full-fat definition of the definite integral. A quick review of two bits of notation:

$\displaystyle\sum_{i=1}^{n} a_n = a_1 + a_2 + a_3 + \cdots + a_n$ is "the sum of a_i for $i = 1$ to n".

$\displaystyle\int_a^b f(x)\,dx =$ the weighted area between $y = f(x)$ and the x-axis for $a \le x \le b$ [area below the x-axis is negative] is "the definite integral of $f(x)$ from a to b".

The following questions all involve the function $f(x) = x^2 - 2x$.

1. Use SageMath to plot $y = f(x)$ for $-1 \le x \le 4$. *[0.5]*

SOLUTION. See the appended notebook. ■

2. Use SageMath to compute the Right-Hand Rule sums approximating the weighted area $\displaystyle\int_0^3 f(x)\,dx$ using n rectangles and the `sum` command, where:

a. $n = 6$ *[1]* **b.** $n = 12$ *[1]* **c.** $n = 18$ *[1]*

SOLUTIONS. The Right-Hand Rule sum for approximating the weighted area $\displaystyle\int_a^b f(x)\,dx$ using n rectangles is $r(n) = \dfrac{b-a}{n}\displaystyle\sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)$. In this case, of course, $a = 0$, $b = 3$, and $f(x) = x^2 - 2x$. For the SageMath part, see the appended notebook. ■

3. Use SageMath to compute the weighted area $\displaystyle\int_0^3 f(x)\,dx$ via the Right-Hand Rule formula, using the `sum` and `limit` commands. *[2]*

SOLUTION. According to the Right-Hand Rule,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right).$$

[*] You should submit your solutions via Blackboard's Assignments module, preferably as a single pdf. If submission via Blackboard fails, please submit your work to your instructor by email or on paper as soon as you can. You may work together, look things up, and use whatever tools you like, so long as you write up your submission by yourself and give due credit to your collaborators and any sources and tools you actually used.

In this case, of course, $a = 0$, $b = 3$, and $f(x) = x^2 - 2x$. For evaluating this limit using SageMath, see the appended notebook. ■

4. Compute $\displaystyle\int_0^3 f(x)\,dx$ via the Right-Hand Rule formula by hand. *[3.5]*

NOTE. You may find the sum formulas $\displaystyle\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ and $\displaystyle\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ useful when doing question **4**.

SOLUTION. As noted above, according to the Right-Hand Rule

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right).$$

In this case, of course, $a = 0$, $b = 3$, and $f(x) = x^2 - 2x$. Here we go:

$$\begin{aligned}
\int_0^3 \left(x^2 - 2x\right)\,dx &= \lim_{n\to\infty} \frac{3-0}{n} \sum_{i=1}^{n} \left(\left[0 + i\frac{3-0}{n}\right]^2 - 2\left[0 + i\frac{3-0}{n}\right]\right) \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(\left[i\frac{3}{n}\right]^2 - 2\left[i\frac{3}{n}\right]\right) = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(\frac{9}{n^2}i^2 - \frac{6}{n}i\right) \\
&= \lim_{n\to\infty} \frac{3}{n} \left[\left(\sum_{i=1}^{n} \frac{9}{n^2}i^2\right) - \left(\sum_{i=1}^{n} \frac{6}{n}i\right)\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \left[\left(\frac{9}{n^2}\sum_{i=1}^{n} i^2\right) - \left(\frac{6}{n}\sum_{i=1}^{n} i\right)\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \left[\left(\frac{9}{n^2} \cdot \frac{n(n+1)(2n+1)}{6}\right) - \left(\frac{6}{n} \cdot \frac{n(n+1)}{2}\right)\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \left[\frac{9}{6} \cdot \frac{(n+1)(2n+1)}{n} - \frac{6}{2} \cdot (n+1)\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \left[\frac{3}{2} \cdot \frac{2n^2 + 3n + 1}{n} - 3(n+1)\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \left[\frac{3}{2}\left(2n + 3 + \frac{1}{n}\right) - 3n - 3\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \left[3n + \frac{9}{2} + \frac{3}{2n} - 3n - 3\right] = \lim_{n\to\infty} \frac{3}{n} \left[\frac{3}{2} + \frac{3}{2n}\right] \\
&= \lim_{n\to\infty} \left[\frac{9}{2n} + \frac{9}{2n^2}\right] = 0 + 0 = 0 \quad \text{Whew!} \quad ■
\end{aligned}$$

5. Use SageMath to compute $\displaystyle\int_0^3 f(x)\,dx$ using the `integral` command. *[1]*

SOLUTION. See the appended notebook. ■

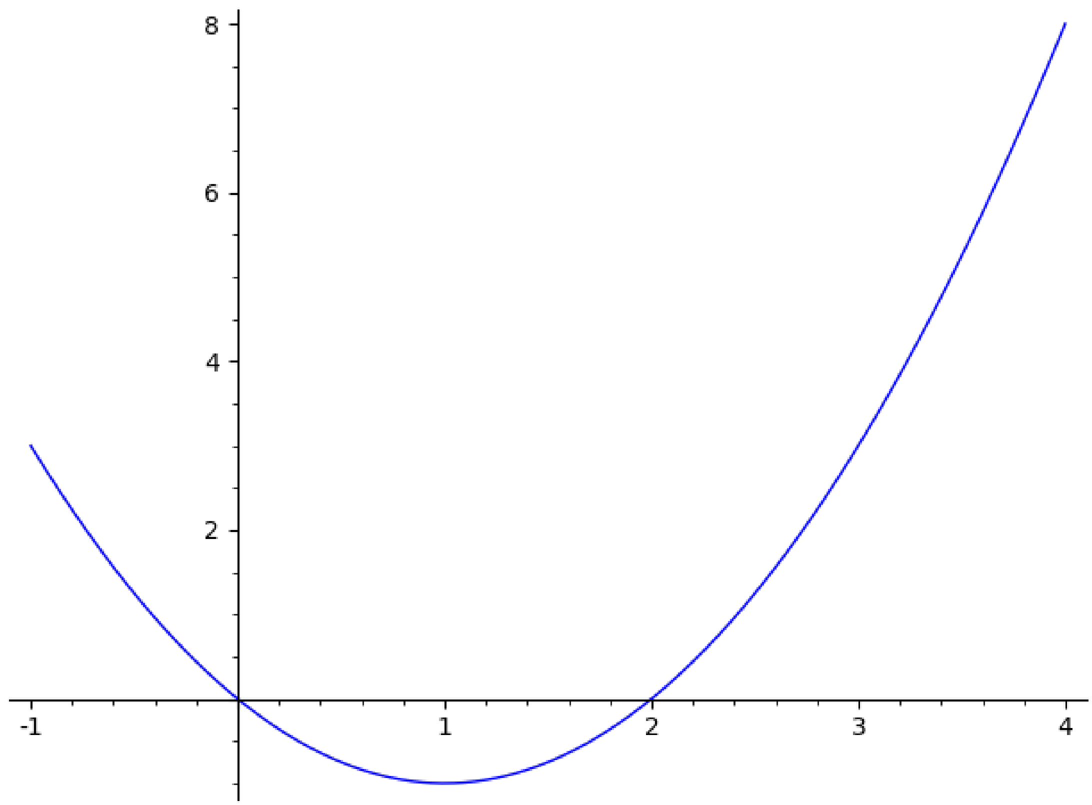
MATH1110H-A5-Solutions-SageMath

November 6, 2025

[1]:
```
# MATH 1110H Assignment #5
# SageMath solutions
```

[2]:
```
# Our function
f = function('f')(x)
f(x) = x^2 - 2*x
# Question 1
plot( f(x), -1, 4 )
```

[2]:



[3]:
```
# A code snippet for computing Right-Hand Rule sums.
var('n')
```

```
var('i')
r = function('r')(n)
a = 0
b = 3
r(n) = (b-a)/n * sum( f(a + i*(b-a)/n), i, 1, n )
```

[4]:
```
# Question 2a
r(6)
```

[4]: 7/8

[5]:
```
# Question 2b
r(12)
```

[5]: 13/32

[6]:
```
# Question 2c
r(18)
```

[6]: 19/72

[7]:
```
# Question 3
limit( r(n), n=oo )
```

[7]: 0

[8]:
```
# Question 5
integral( f(x), x, 0, 3 )
```

[8]: 0

[ ]:
```

```