MATH1110H-A3-Solutions

October 1, 2025

[1]:
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# MATH 1110H,  Fall 2025
# Solutions to Assignment #3
#
# The solution to 1c and 2d, the parts that need to be done by hand,
# are appended to the pdf version of this notebook.
```

[2]:
```
# 1a
var('x')
# Declare y to be a (currently unknown) function of the variable x.
y = function('y')(x)
# Name the differential equation, to avoid writing it out a lot.
DE = (1/y^2)*diff(y,x) == e^(-x) - e^x
# Solve it for y!
desolve( DE, y )
```

[2]:
```
(1/y(x)) == (e^(2*x) + 1)*e^(-x) + _C
```

[3]:
```
# Sigh. SageMath actually solves for 1/y.  We can rearrange this by
# hand to solve for y,  or we can do this using the solve command.
# To avoid error messages that "name _C is not defined",  we first
# declare c to be anew variable and use it in place of _C.
var('c')
solve( (1/y(x)) == (e^(2*x) + 1)*e^(-x) + c, y )
```

[3]:
```
[y(x) == e^x/(c*e^x + e^(2*x) + 1)]
```

[4]:
```
# 1b
desolve( DE, y, ics=[0,0.5] )
```

[4]:
```
(1/y(x)) == (e^(2*x) + 1)*e^(-x)
```

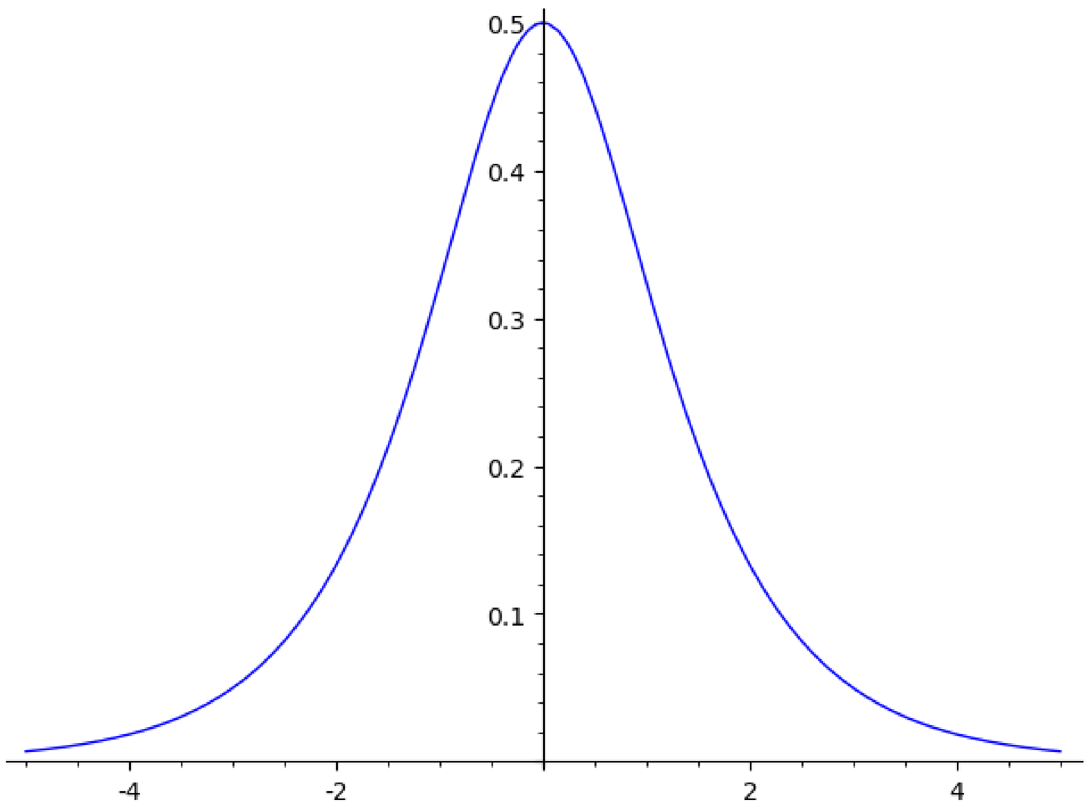
[5]:
```
# We resort to the solve command again to solve for y.
solve( (1/y(x)) == (e^(2*x) + 1)*e^(-x), y )
```

[5]:
```
[y(x) == e^x/(e^(2*x) + 1)]
```

[6]:
```
# 1c
# Appended to the pdf version of this notebook.
```

[7]:
```
# 2a
clear_vars()  # Since we'll be using y, in particular, as a variable,
var('x')      # rather than as a function.
var('y')
f = function('f')(x)  # We'll call the function f(x) instead.
f(x) = e^x/(e^(2*x) + 1)
plot( f(x), -5, 5 )
```

[7]:



[8]:
```
# 2b
# Looking at the plot,  we can invert the part of f(x) for x<=0
# and the part of f(x) for x>=0,  since each of these assigns
# an unique y for x.  We can't invert any part that straddles 0
# because then there would be different x's which give the same
# y value,  i.e. f(x) is not 1-1 on any interval that includes
# both positive and negative x's.
```

[9]:
```
# 2c
solve( x == f(y), y )  # Following the hint...
```

[9]:
```
[y == log(-1/2*sqrt(-4*x^2 + 1)/x + 1/2/x), y == log(1/2*sqrt(-4*x^2 + 1)/x +
1/2/x)]
```

[10]:
```
# Note that we get two possibilities for the inverse of f(x).
# The first inverts the part of y = f(x) with x <= 0 and the
# second the part of y = f(x) with x >= 0.  [How can we tell?]
```

[11]:
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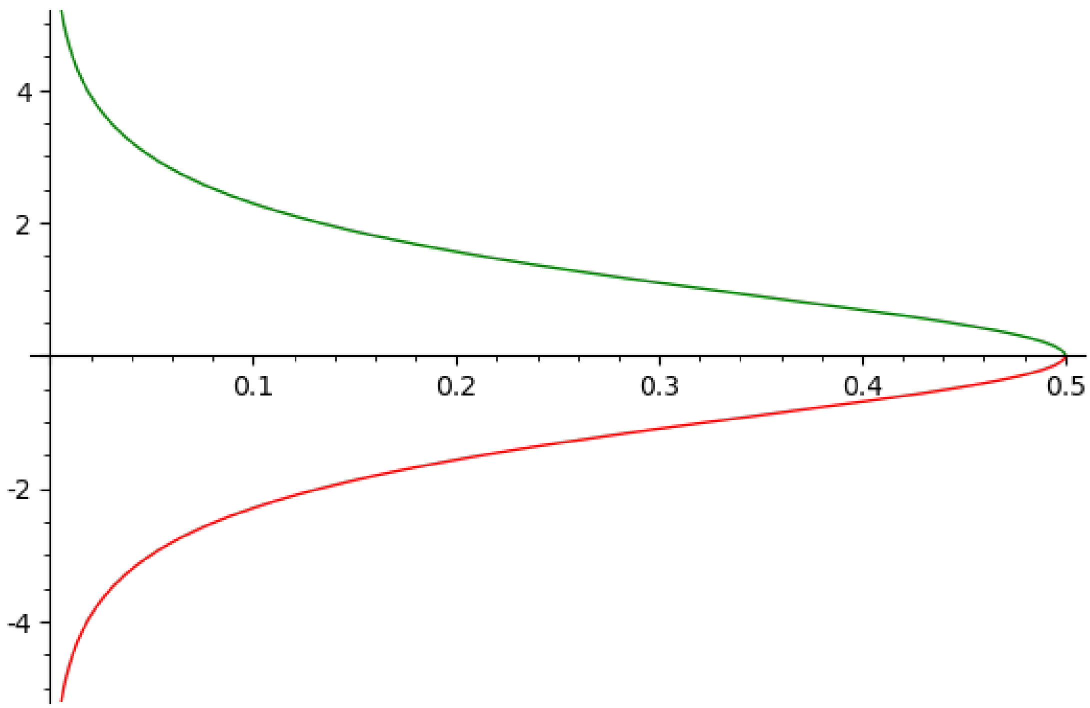
# Well,  we can plot them...  This is just for fun!
p1 = plot( log(-1/2*sqrt(-4*x^2 + 1)/x + 1/2/x), 0, 0.5, ymin=-5, ymax=5,␣
 ↪color='red')
p2 = plot( log(1/2*sqrt(-4*x^2 + 1)/x + 1/2/x), 0, 0.5, ymin=-5, ymax=5,␣
 ↪color='green')
p1 + p2
```

[11]:



[12]:
```
# 2d
# Appended to the pdf version of this notebook.
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1c. Check by hand that the solution you obtained in **b** satisfies the given differential equation and the given initial condition. *[1]*

SOLUTION. The solution obtained in part **b** is $y = \frac{e^x}{e^{2x}+1}$. When $x = 0$, this gives $y = \frac{e^0}{e^{2\cdot 0}+1} = \frac{1}{1+1} = \frac{1}{2}$, so this solution satisfies the given initial condition. To check that it satisfies the differential equation, we first need to differetiate it:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x}{e^{2x}+1}\right) = \frac{\left[\frac{d}{dx}e^x\right]\left(e^{2x}+1\right) - e^x\left[\frac{d}{dx}\left(e^{2x}+1\right)\right]}{\left(e^{2x}+1\right)^2} \quad \text{[Quotient Rule]}$$

$$= \frac{\left[e^x\right]\left(e^{2x}+1\right) - e^x\cdot\left[2e^{2x}+0\right]}{\left(e^{2x}+1\right)^2} = \frac{e^{3x}+e^x-2e^{3x}}{\left(e^{2x}+1\right)^2} = \frac{e^x-e^{3x}}{\left(e^{2x}+1\right)^2}$$

We plug $\frac{dy}{dx}$ and y into the left-hand side of the given differential equation, $\frac{1}{y^2}\cdot\frac{dy}{dx} = e^{-x}-e^x$, and hope that the right-hand side appears when we simplify ...

$$\frac{1}{y^2}\cdot\frac{dy}{dx} = \frac{1}{\left(\frac{e^x}{e^{2x}+1}\right)^2}\cdot\frac{e^x-e^{3x}}{\left(e^{2x}+1\right)^2} = \left(\frac{e^{2x}+1}{e^x}\right)^2\cdot\frac{e^x-e^{3x}}{\left(e^{2x}+1\right)^2} = \frac{\left(e^{2x}+1\right)^2}{\left(e^x\right)^2}\cdot\frac{e^x-e^{3x}}{\left(e^{2x}+1\right)^2}$$

$$= \frac{e^x-e^{3x}}{e^{2x}} = e^{-x}-e^x \quad \text{as required.}$$

Thus $y = \frac{e^x}{e^{2x}+1}$ also satisfies the given differential equation. ■

2d. Find (the possibilities for) $f^{-1}(x)$ by hand, showing all the principal steps. *[2]*

HINT. $y = f^{-1}(x) \iff x = f(y)$.

SOLUTION. Following the hint, since $f(x) = \frac{e^x}{e^{2x}+1}$, we find $f^{-1}(x)$ by solving $x = f(y) = \frac{e^y}{e^{2y}+1}$ for y in terms of x.

$$x = \frac{e^y}{e^{2y}+1} \implies x\left(e^{2y}+1\right) = e^y \implies xe^{2y}+x-e^y = 0 \implies x\left(e^y\right)^2 - e^y + x = 0$$

We now apply the quadratic formula to the last equation to solve for e^y.

$$x\left(e^y\right)^2 - e^y + x = 0 \implies e^y = \frac{-(-1)\pm\sqrt{(-1)^2-4x\cdot x}}{2x} = \frac{1\pm\sqrt{1-4x^2}}{2x}$$

Finally, we use the natural logarithm function to isolate y.

$$y = \ln\left(e^y\right) = \ln\left(\frac{1\pm\sqrt{1-4x^2}}{2x}\right)$$

This is a more compact form of the solutions obtained in part **2c** using SageMath, but it's not too hard to check that they are actually the same. ■