Mathematics 1110H (Section B) – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Fall 2023

Solutions to Quiz #10
Some Indefinite Integrals

REMINDER. While you are allowed to work together and look things up when doing the quizzes and assignments, your submission should be written up entirely by yourself, giving credit to any collaborators or sources that you ended up actually using. *Please show all your steps and simplify your answers as far as practical.*

Work out each of the following indefinite integrals.

1. $\displaystyle\int \frac{x}{\sqrt{9-x^2}}\,dx$. *[1]*

SOLUTION. We'll go for the "whole hog" substitution $u = \sqrt{9-x^2}$, so

$$\frac{du}{dx} = \frac{d}{dx}\sqrt{9-x^2} = \frac{1}{2\sqrt{9-x^2}} \cdot \frac{d}{dx}\left(9-x^2\right) = \frac{-2x}{2\sqrt{9-x^2}} = \frac{-x}{\sqrt{9-x^2}},$$

and thus $du = \dfrac{-x}{\sqrt{9-x^2}}\,dx$. Except for the negative sign, this is the integrand; we multiply on both side by -1 to get $\dfrac{x}{\sqrt{9-x^2}}\,dx = (-1)\,du$. It now follows that:

$$\int \frac{x}{\sqrt{9-x^2}}\,dx = \int (-1)\,du = -u + C = -\sqrt{9-x^2} + C \quad \square$$

2. $\displaystyle\int 16x^3 \ln(x)\,dx$. *[1]*

SOLUTION. Since we have a product of two dissimilar functions in the integrand and there is no obvious substitution to simplify matters, we will try integration by parts. Since it is easier to differentiate $\ln(x)$ than integrate it, we'll use the parts $u = \ln(x)$ and $v' = 16x^3$, so $u' = \dfrac{1}{x}$ and $v = 16 \cdot \dfrac{x^4}{4} = 4x^4$. It now follows that:

$$\begin{aligned}\int 16x^3 \ln(x)\,dx &= 4x^4\ln(x) - \int \frac{1}{x}\cdot 4x^4\,dx = 4x^4\ln(x) - \int 4x^3\,dx \\ &= 4x^4\ln(x) - 4\cdot\frac{x^4}{4} + C = 4x^4\ln(x) - x^4 + C \quad \square\end{aligned}$$

3. $\displaystyle\int \tan(x)\,dx$. *[1]*

SOLUTION. Recall that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$. We'll use the substitution $w = \cos(x)$, so $\dfrac{dw}{dx} = \dfrac{d}{dx}\cos(x) = -\sin(x)$. Thus $dw = -\sin(x)\,dx$ and $\sin(x)\,dx = (-1)\,dw$. It now follows that:

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = \int \frac{1}{w}(-1)\,dw = -\ln(w) + C = -\ln\left(\cos(x)\right) + C \quad \square$$

4. $\displaystyle\int 6\sin\left(\cos^2(x)\right)\cos(x)\sin(x)\,dx$. *[1]*

SOLUTION. The $\cos^2(x)$ shoved inside the first sin in the integrand, so we will try to substitute for it in the hope of simplifying the integrand down to something manageable. If we let $z = \cos^2(x)$, then

$$\frac{dz}{dx} = \frac{d}{dx}\cos^2(x) = 2\cos(x)\frac{d}{dx}\cos(x) = 2\cos(x)\left(-\sin(x)\right) = -2\cos(x)\sin(x),$$

so $dz = -2\cos(x)\sin(x)\,dx$ and $(-1)\,dz = 2\cos(x)\sin(x)\,dx$. By an amazing coincidence, we have $6\cos(x)\sin(x) = 3\cdot 2\cos(x)\sin(x)$ available to us in the rest of the integrand. It now follows that:

$$\begin{aligned}\int 6\sin\left(\cos^2(x)\right)\cos(x)\sin(x)\,dx &= \int 3\sin(z)(-1)\,dz = 3\int\left(-\sin(z)\right)\,dz \\ &= 3\cos(z) + C = 3\cos\left(\cos^2(x)\right) + C \quad \square\end{aligned}$$

5. $\displaystyle\int x^2\sinh(x)\,dx$. *[1]*

Hint: This is the hyperbolic function $\sinh(x)$, not the trigonometeric function $\sin(x)$. Recall that $\sinh(x)$ and $\cosh(x)$ are each other's derivatives, without any negative signs.

SOLUTION. Since we have a product of two dissimilar functions in the integrand and there is no obvious substitution to simplify matters, we will try integration by parts. Since positive powers of x reduce when differentiated and increase when integrated, while sinh and cosh just trade with each other whether differentiated or integrated, we will try the parts $u = x^2$ and $v' = \sinh(x)$, so $u' = 2x$ and $v = \cosh(x)$. This gives:

$$\int x^2\sinh(x)\,dx = x^2\cosh(x) - \int 2x\cosh(x)\,dx = x^2\cosh(x) - 2\int x\cosh(x)\,dx$$

To resolve the remaining integral, we will use parts gain, this time with $s = x$ and $t' = \cosh(x)$, so $s' = 1$ and $t = \sinh(x)$. This gives:

$$\int x\cosh(x)\,dx = x\sinh(x) - \int 1\sinh(x)\,dx = x\sinh(x) - \cosh(x)$$

We'll put in the generic constant at the next step to avoid later multiplying by 2. It now follows that:

$$\begin{aligned}\int x^2\sinh(x)\,dx &= x^2\cosh(x) - 2\int x\cosh(x)\,dx \\ &= x^2\cosh(x) - 2\left[x\sinh(x) - \cosh(x)\right] + C \\ &= x^2\cosh(x) - 2x\sinh(x) + 2\cosh(x) + C \quad \square\end{aligned}$$