Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals

Trent University, Fall 2023

Assignment #4
Chase and Drag

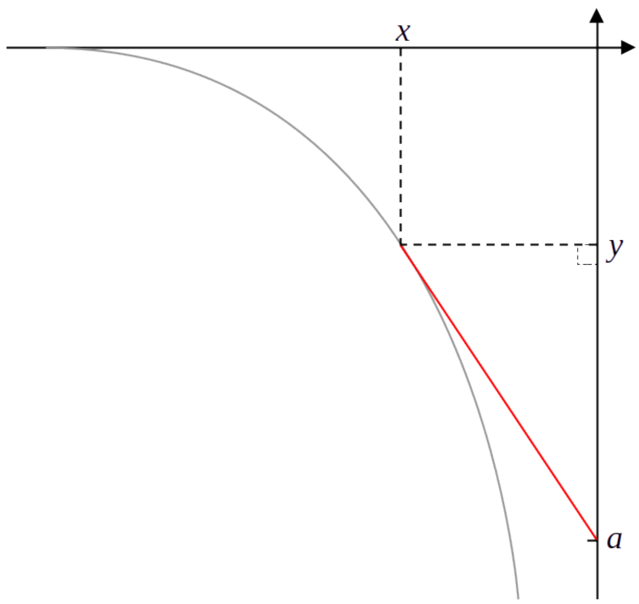
Due[*] just before midnight on Friday, 10 November.

Minimax is walking Big Dog in the Cartesian plane, with the leash between them at its full 10 *cubit*[†] They walk along the negative x-axis, moving towards the origin, but just as Big Dog reaches the origin a Showy Squirrel is spotted far down the negative y-axis. Big Dog starts running down the negative y-axis towards the Showy Squirrel, dragging Minimax along. The leash stays straight and at full extension throughout and at any given instant the leash is tangent to the curve Minimax is dragged along.

1. If (x, y) is a point on the curve Minimax is being dragged along, find $\dfrac{dy}{dx}$ as a function of x. *[5]*

Hint: If Big Dog is at $(0, a)$ on the y-axis at the instant that Minimax is (x, y), then the distance between them is 10, but can also be computed another way. Similarly, the slope of the curve Minimax is being dragged along can be computed in two different ways at that instant.

Solution. Following the hint, here is a diagram of the situation when Minimax is at (x, y) and Big Dog is at $(0, a)$:



[*] You should submit your solutions via Blackboard's Assignments module, preferably as a single pdf. If submission via Blackboard fails, please submit your work to your instructor by email or on paper.

[†] A cubit is an unit of length based on the length of the forearm from the elbow to the tip of the middle finger. Cubits – of various lengths! – were commonly used in ancient times in the "Fertile Crescent" running from Mesopotamia at one end to Egypt at the other, and in adjacent regions.

Consider the triangle formed by the points (x, y), $(0, y)$, and $(0, a)$; note that all three of x, y, and a are negative. It is a right triangle with short sides of length $0 - x = -x$ and $y - a$, and with a hypotenuse of length 10, since the leash coincides with the hypotenuse, *i.e.* the line segment joining (x, y) [Minimax] to $(0, a)$ [Big Dog]. By the Pythagorean Theorem, it follows that $(-x)^2 + (y - a)^2 = 10^2$. We can use this fact to solve for $a - y$ in terms of x, which we will need shortly:

$$\begin{aligned}(-x)^2 + (y-a)^2 = 10^2 &\implies x^2 + (y-a)^2 = 100 \implies (y-a)^2 = 100 - x^2\\ &\implies y - a = \sqrt{100 - x^2} \quad \text{(Positive root since } y > a.)\\ &\implies a - y = -\sqrt{100 - x^2}\end{aligned}$$

Now consider the slope of the line joining (x, y) to $(0, a)$. Since the leash is part of this line and we are told the leash is tangent to the curve Minimax is being dragged along athe point (x, y), the slope of the line is given by the derivative $\frac{dy}{dx}$. On the other hand, the slope of the line can be computed using the fact that it joins the points (x, y) and $(0, a)$:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{a - y}{0 - x} = \frac{-\sqrt{100 - x^2}}{-x} = \frac{\sqrt{100 - x^2}}{x}$$

It follows that $\frac{dy}{dx} = \frac{\sqrt{100 - x^2}}{x}$, so we have found $\frac{dy}{dx}$ as a function of x. □

2. Use SageMath to solve the differential equation you obtained in solving **1**. (Initial conditions included!) *[3]*

SOLUTION. Observe that because Minimax is at $(-10, 0)$ at the start, we have initial conditions for the differential equation $\frac{dy}{dx} = \frac{\sqrt{100 - x^2}}{x}$ obtained in **1**, namely that $y = 0$ when $x = -10$. We plug the differential equation with these initial conditions into `desolve` in SageMath:

```
[1]: y = function('y')(x)
     desolve(diff(y,x) == sqrt(100-x^2)/x, y, ics=[-10,0])

[1]: sqrt(-x^2 + 100) + 10*log(20) - 10*log(20*(sqrt(-x^2 + 100) + 10)/abs(x))
```

Thus the equation of the curve Minimax is dragged along is:

$$y = \sqrt{100 - x^2} + 10\ln(20) - 10\ln\left(\frac{20\sqrt{100 - x^2} + 10)}{|x|}\right)$$

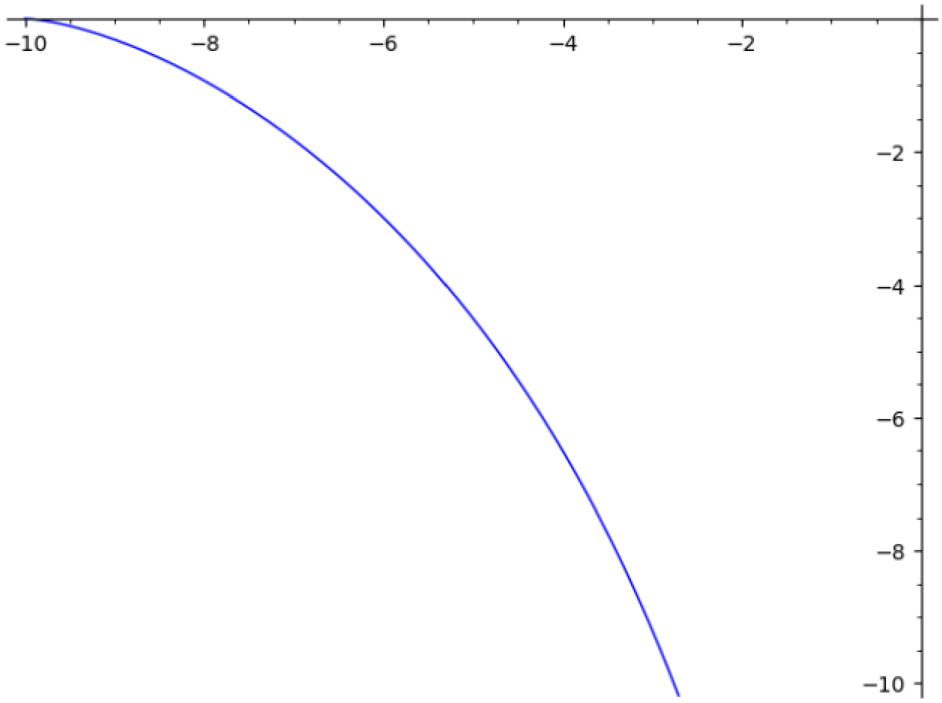
This could be rearranged in all sorts of ways, some of which are superficially simpler. □

3. Use SageMath to plot the curve Minimax is dragged along for $-10 \leq y \leq 0$. *[1]*

SOLUTION. Another job for the basic `plot` command, made easier by copy-pasting the solution given by SageMath to the previous problem:

```
[2]: plot(sqrt(-x^2 + 100) + 10*log(20) - 10*log(20*(sqrt(-x^2 + 100) + 10)/abs(x)),
     -10, 0, ymin=-10, ymax=0)
```

[2]:



$\square$

4. Assuming Big Dog keeps running down the y-axis and continues to drag Minimax, does Minimax ever reach the y-axis? Explain why or why not. *[1]*

SOLUTION. We better hope that Minimax never reaches the y-axis along the curve we obtained in **2**, for the y-axis is the line $x = 0$ and the function defining the curve has a component where we divide by $|x|$ ... :-) $\square$