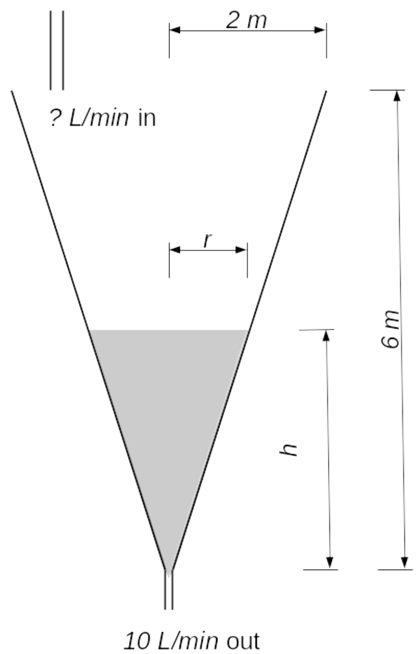
Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals (Section C)
TRENT UNIVERSITY, Fall 2021

Solution to Quiz #7
Wednesday, 10 November.

Do the following problem.



1. A conical tank 6 m tall and with a radius of 2 m at the blunt end is oriented point down and blunt end up. A pipe feeds water into the tank at the top at an unknown constant rate, while another pipe at the tip of the tank removes water from the tank at a rate of 10 L/min. Suppose the water level in the tank is rising at a rate of ~~20 cm~~ 20 cm/min when the depth of the water in the tank is 2 m. Find the rate at which water is being fed into the tank. *[5]*

SOLUTION. Note that the water in the tank, at any given instant, occupies a volume shaped like a cone with the same proportions as the tank. Let h be the depth of the water in the tank and r be the radius at the top of the conical shape the water occupies. Then $h/r = 6/2 = 3$, so $h = 3r$. The volume of a cone with height h and radius r is $V = \dfrac{\pi r^2 h}{3}$, so the volume of the water in the tank is given by $V = \dfrac{\pi r^2 h}{3} = \dfrac{\pi r^2 3r}{3} = \pi r^3$.

Let's review the given information with the formulas in the above paragraph in mind. Denote by x the constant rate at which water flows into the tank from the pipe at the top; we'll measure this in m^3/min. The constant rate at wihich the tank is being drained by the bottom pipe is 10 $L/min = 0.01\ m^3/min$, so the rate of change of the volume of the water in the tank must be $\dfrac{dV}{dt} = x - 0.01\ m^3/min$. On the other hand, calculus tells us that the rate of change of the volume of the water in the tank at any given instance should be:

$$\frac{dV}{dt} = \frac{d}{dt}\pi r^3 = \pi\frac{dr^3}{dr} \cdot \frac{dr}{dt} = 3\pi r^2 \frac{dr}{dt}$$

We are also told that at the instant that the depth of the water in the tank is 2 m, *i.e.* when $h = 2\ m$, it is rising at a rate of 20 cm/min *i.e.* at a rate of $\frac{dh}{dt} = 0.2\ m/min$. Since $h = 3r$, $\frac{dh}{dt} = 3\frac{dr}{dt}$, so it follows that at the instant that $h = 2\ m$, we have $r = \frac{2}{3} \approx 0.6667$ and $\frac{dr}{dt} = \frac{0.2}{3} = \frac{1}{15} \approx 0.0667\ m/min$. This, in turn, means that when $h = 2\ m$, the rate of change of volume is given by:

$$\left.\frac{dV}{dt}\right|_{h=2} = 3\pi r^2 \frac{dr}{dt} = 3\pi \left(\frac{2}{3}\right)^2 \frac{1}{15} = \frac{3\pi \cdot 4}{9 \cdot 15} = \frac{4\pi}{45} \approx 0.2793\ m^3/min$$

Thus, at the instant that $h = 2\ m$, we have $x - 0.01 = \frac{dV}{dt} = \frac{4\pi}{45}\ m^3/min$, so the rate x at which water flows in through the pipe at the top must be $x = \frac{4\pi}{45} + 0.01 \approx 0.2893\ m^2/min = 289.3\ L/min$. ■

[Total = 5]