Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals (Section C)
TRENT UNIVERSITY, Fall 2021

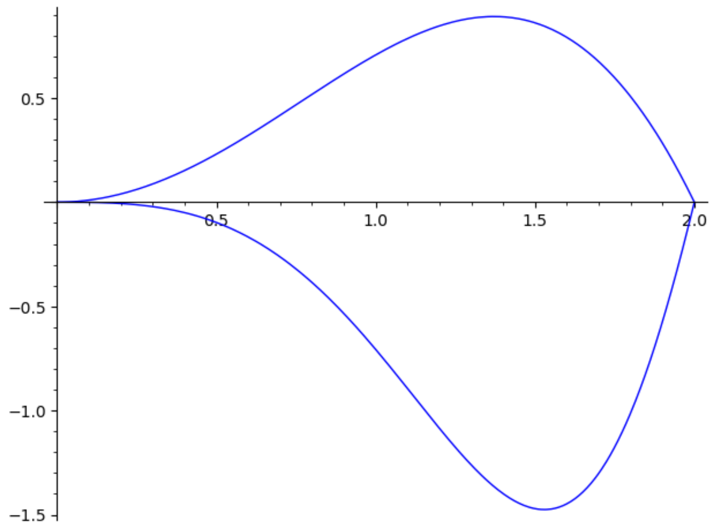
Quiz #10
Wednesday, 1 December.

1. Find the area between the curves $y = x^2 \cos\left(\frac{\pi x}{4}\right)$ and $y = -x \sin\left(\frac{\pi x^2}{4}\right)$, where $0 \le x \le 2$. *[5]*

SOLUTION. *(By hand and calculator.)* To get some idea of what the region looks like, it is a good idea to plot the two curves on the given interval. Recall that we may use any calculator we like on the quizzes. Stretching the notion of "calculator" just a little, typing

```
plot(x^2*cos(pi*x/4),0,2)+plot(-x*sin(pi*x^2/4),0,2)
```

into `SageMath` gives the following picture:



This suggests the two curves intersect at $(0,0)$ and $(2,0)$, and that between these points $y = x^2 \cos\left(\frac{\pi x}{4}\right)$ is positive and $y = -x \sin\left(\frac{\pi x^2}{4}\right)$ is negative. We can verify the intersections by observing that when $x = 0$, $0^2 \cos\left(\frac{\pi \cdot 0}{4}\right) = 0 \cdot 1 = 0 = -0 \cdot 0 = -0 \sin\left(\frac{\pi \cdot 0^2}{4}\right)$, and that when $x = 2$, $2^2 \cos\left(\frac{\pi \cdot 2}{4}\right) = 4 \cdot 0 = 0 = -2 \cdot 0 = -2 \sin\left(\frac{\pi \cdot 2^2}{4}\right)$. As for the rest, $x^2 > 0$ for all $x \ne 0$, and $0 < \frac{\pi x}{4} < \frac{\pi}{2}$ when $0 < x < 2$, so $\cos\left(\frac{\pi x}{4}\right) > 0$, and it follows that $y = x^2 \cos\left(\frac{\pi x}{4}\right) > 0$ for $0 < x < 2$. Similarly, when $0 < x < 2$, $-x < 0$ and $0 < \frac{\pi x^2}{4} < \pi$, so $\sin\left(\frac{\pi x^2}{4}\right) > 0$, from which it follows that $y = -x \sin\left(\frac{\pi x^2}{4}\right) < 0$ for $0 < x < 2$.

It follows that the area between the two curves for $0 \leq x \leq 2$ is given by:

$$\begin{aligned}
A &= \int_0^2 \left(x^2 \cos\left(\frac{\pi x}{4}\right) - \left(-x \sin\left(\frac{\pi x^2}{4}\right)\right)\right) dx \\
&= \int_0^2 \left(x^2 \cos\left(\frac{\pi x}{4}\right) + x \sin\left(\frac{\pi x^2}{4}\right)\right) dx \\
&= \int_0^2 x^2 \cos\left(\frac{\pi x}{4}\right) dx + \int_0^2 x \sin\left(\frac{\pi x^2}{4}\right) dx
\end{aligned}$$

We will use integration by parts to compute the former integral and substitution to compute the latter integral.

For the former integral, we will use the parts $u = x^2$ and $v' = \cos\left(\frac{\pi x}{4}\right)$, so $u' = 2x$ and $v = \frac{4}{\pi}\sin\left(\frac{\pi x}{4}\right)$. (The problem of integrating $\cos\left(\frac{\pi x}{4}\right)$ is left to the reader; try the substitution $t = \frac{\pi x}{4}$, so $dt = \frac{\pi}{4}\,dx$ and $dx = \frac{4}{\pi}\,dt$.)

$$\begin{aligned}
\int_0^2 x^2 \cos\left(\frac{\pi x}{4}\right) dx &= x^2 \cdot \frac{4}{\pi} \sin\left(\frac{\pi x}{4}\right)\Bigg|_0^2 - \int_0^2 2x \cdot \frac{4}{\pi}\sin\left(\frac{\pi x}{4}\right) dx \\
&= \frac{16}{\pi}\sin\left(\frac{\pi}{2}\right) - 0\sin(0) - \frac{8}{\pi}\int_0^2 x \cdot \sin\left(\frac{\pi x}{4}\right) dx \\
&= \frac{16}{\pi} \cdot 1 - 0 - \frac{8}{\pi}\int_0^2 x \cdot \sin\left(\frac{\pi x}{4}\right) dx \\
&= \frac{16}{\pi} - \frac{8}{\pi}\int_0^2 x \cdot \sin\left(\frac{\pi x}{4}\right) dx
\end{aligned}$$

This requires us to use parts again, this time with $u = x$ and $v' = \sin\left(\frac{\pi x}{4}\right)$, so $u' = 1$ and $v = -\frac{4}{\pi}\cos\left(\frac{\pi x}{4}\right)$. (The problem of integrating $\sin\left(\frac{\pi x}{4}\right)$ is also left to the reader.) Thus:

$$\begin{aligned}
\int_0^2 x^2 \cos\left(\frac{\pi x}{4}\right) dx &= \frac{16}{\pi} - \frac{8}{\pi}\int_0^2 x \cdot \sin\left(\frac{\pi x}{4}\right) dx \\
&= \frac{16}{\pi} - \frac{8}{\pi}\left[x \cdot \left(-\frac{4}{\pi}\cos\left(\frac{\pi x}{4}\right)\right)\Bigg|_0^2 - \int_0^2 1 \cdot \left(-\frac{4x}{\pi}\cos\left(\frac{\pi x}{4}\right)\right) dx \right] \\
&= \frac{16}{\pi} - \frac{8}{\pi}\left[-\frac{4x}{\pi}\cos\left(\frac{\pi x}{4}\right)\Bigg|_0^2 + \frac{4}{\pi}\int_0^2 \cos\left(\frac{\pi x}{4}\right) dx \right] \\
&= \frac{16}{\pi} + \frac{32x}{\pi^2}\cos\left(\frac{\pi x}{4}\right)\Bigg|_0^2 - \frac{32}{\pi^2}\int_0^2 \cos\left(\frac{\pi x}{4}\right) dx \\
&= \frac{16}{\pi} + \frac{32 \cdot 2}{\pi^2}\cos\left(\frac{\pi}{2}\right) - \frac{32 \cdot 0}{\pi^2}\cos(0) - \frac{32}{\pi^2}\int_0^2 \cos\left(\frac{\pi x}{4}\right) dx \\
&= \frac{16}{\pi} + \frac{64}{\pi^2} \cdot 0 - 0 \cdot (-1) - \frac{32}{\pi^2}\int_0^2 \cos\left(\frac{\pi x}{4}\right) dx \\
&= \frac{16}{\pi} - \frac{32}{\pi^2}\int_0^2 \cos\left(\frac{\pi x}{4}\right) dx
\end{aligned}$$

The problem of finding the antiderivative of $\cos\left(\frac{\pi x}{4}\right)$ was considered, and left to the reader, above. Plugging in what it was claimed to be yields:

$$\begin{aligned}
\int_0^2 x^2 \cos\left(\frac{\pi x}{4}\right)\,dx &= \frac{16}{\pi} - \frac{32}{\pi^2}\int_0^2 \cos\left(\frac{\pi x}{4}\right)\,dx \\
&= \frac{16}{\pi} - \frac{32}{\pi^2}\cdot\frac{4}{\pi}\sin\left(\frac{\pi x}{4}\right)\Big|_0^2 \\
&= \frac{16}{\pi} - \frac{128}{\pi^3}\sin\left(\frac{\pi x}{4}\right)\Big|_0^2 \\
&= \frac{16}{\pi} - \left(\frac{128}{\pi^3}\sin\left(\frac{\pi}{2}\right) - \frac{128}{\pi^3}\sin(0)\right) \\
&= \frac{16}{\pi} - \left(\frac{128}{\pi^3}\cdot 1 - \frac{128}{\pi^3}\cdot 0\right) \\
&= \frac{16}{\pi} - \frac{128}{\pi^3}
\end{aligned}$$

For the latter integral, we will use the substitution $w = \frac{\pi x^2}{4}$, so $dw = \frac{\pi x}{2}\,dx$ and $x\,dx = \frac{2}{\pi}\,dw$, and change the limits as we go along: $\begin{matrix} x & 0 & 2 \\ w & 0 & \pi \end{matrix}$

$$\begin{aligned}
\int_0^2 x\sin\left(\frac{\pi x^2}{4}\right)\,dx &= \int_0^\pi \sin(w)\,\frac{2}{\pi}\,dw = \frac{2}{\pi}\left(-\cos(w)\right)\Big|_0^\pi \\
&= \frac{2}{\pi}\left(-\cos(\pi)\right) - \frac{2}{\pi}\left(-\cos(0)\right) = \frac{2}{\pi}\left(-(-1)\right) - \frac{2}{\pi}(-1) \\
&= \frac{2}{\pi} + \frac{2}{\pi} = \frac{4}{\pi}
\end{aligned}$$

Thus the area of the region between the two curves is:

$$\begin{aligned}
A &= \int_0^2 x^2\cos\left(\frac{\pi x}{4}\right)\,dx + \int_0^2 x\sin\left(\frac{\pi x^2}{4}\right)\,dx \\
&= \left(\frac{16}{\pi} - \frac{128}{\pi^3}\right) + \frac{4}{\pi} = \frac{20}{\pi} - \frac{128}{\pi^3} \approx 2.2380 \qquad \blacksquare
\end{aligned}$$

CHECK. Using the `SageMath` "calculator" to evaluate the area integral lets us check our work. Typing

```
integral(x^2*cos(pi*x/4)+x*sin(pi*x^2/4),x,0,2)
```

into `SageMath` gives the output

```
4*(5*pi^2 - 32)/pi^3
```

which looks a little different from what was obtained by hand. Being too lazy to actually do algebra to check we use SageMath again to work out the decimal to check. Typing in

```
N(4*(5*pi^2 - 32)/pi^3)
```

yields

```
2.23800131622628
```

so we probably got it right.