Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals (Section C)
Trent University, Fall 2021

Solutions to Quiz #1
Wednesday, 22 September.

Do all three of the following problems.

1. Use the ε–δ definition of limits to verify that $\lim_{x \to 2} (2x-5) = -1$. You may use either the standard version or the game version of the ε–δ definition of limits. *[2]*

Solution the First. We will use the standard ε–δ definition of limits in this solution. We need to show that given any $\varepsilon > 0$, we can find a $\delta > 0$ such that if $|x-2| < \delta$, then $|(2x-5)-(-1)| < \varepsilon$. As usual, we try to reverse-engineer the necessary δ by working backward from the desired conclusion that $|(2x-5)-(-1)| < \varepsilon$. Here goes:

$$\begin{aligned} |(2x-5)-(-1)| < \varepsilon &\iff |2x-5+1| < \varepsilon \iff |2x-4| < \varepsilon \\ &\iff |2(x-2)| < \varepsilon \iff 2\,|x-2| < \varepsilon \\ &\iff |x-2| < \frac{\varepsilon}{2} \end{aligned}$$

Now let $\delta = \frac{\varepsilon}{2}$. Since every step in the process above is reversible, if we have $|x-2| < \delta = \frac{\varepsilon}{2}$, we must also get that $|(2x-5)-(-1)| < \varepsilon$, as required by the definition. Thus $\lim_{x \to 2} (2x-5) = -1$ according to the standard version of the ε–δ definition of limits. $\square$

Solution the Second. We will use the alternate version of the ε–δ definition of limits in this solution. To verify that $\lim_{x \to 2} (2x-5) = -1$ using this version of the definition, we need to find a winning strategy for player B in the corresponding limit game. Recall that there are three moves in the game:

1. Player A chooses an $\varepsilon > 0$.
2. Player B then chooses a $\delta > 0$.
3. Player A then chooses an x with the restriction that $|x - -2| < \varepsilon$.

   Player A wins the game if $|(2x-5)-(-1)| \geq \varepsilon$ and Player B wins the game if $|(2x-5)-(-1)| < \varepsilon$.

A winning strategy for player B must therefore be a method for picking a $\delta > 0$ in response to player A's choice of $\varepsilon > 0$ that ensures that no matter how player A may try to choose an x with $|x-2| < \delta$, player B wins, *i.e.* $|(2x-5)-(-1)| < \varepsilon$. As in the solution using the standard definition, we reverse-engineer how to choose the δ by working backwards from what player B needs to to win:

$$\begin{aligned} |(2x-5)-(-1)| < \varepsilon &\iff |2x-5+1| < \varepsilon \iff |2x-4| < \varepsilon \\ &\iff |2(x-2)| < \varepsilon \iff 2\,|x-2| < \varepsilon \\ &\iff |x-2| < \frac{\varepsilon}{2} \end{aligned}$$

[Yup! It's exactly the same process ... basically, because it's really the same problem.] This suggests that having player B respond to player A's choice of $\varepsilon > 0$ by playing $\delta = \varepsilon/2$ ought to win the game for player B. Let's see:

If player A plays $\varepsilon > 0$ and player B responds by playing $\delta = \varepsilon/2$, then player A must respond in turn with an x such that $|x-2| < \delta = \frac{\varepsilon}{2}$. As every step of the reverse-engineering process is reversible, it must the follow that $|(2x-5)-(-1)| < \varepsilon$, which means that B wins.

It follows that hplaying $\delta = \varepsilon/2$ in response to player A's choice of $\varepsilon > 0$ is a winning strategy for player B. since B has winning strategy in the corresponding limit game, it follows by the alternate version of the ε–δ definition of limits that $\lim_{x\to 2}(2x-5) = -1$. ■

2. Using the practical rules for computing limits, find $\displaystyle\lim_{x\to -3} \frac{x^4-81}{x^2-9}$. *[1.5]*

Solution. The given limit is apparently indeterminate since $x^4 - 81 \to 0$ and $x^2 - 9 \to 0$ as $x \to -3$. Fortunately, the denominator is a factor of the numerator, $x^4 - 81 = \left(x^2-9\right)\left(x^2+9\right)$, so we can compute the limit after a little cancellation:

$$\lim_{x\to -3} \frac{x^4-81}{x^2-9} = \lim_{x\to -3} \frac{\left(x^2-9\right)\left(x^2+9\right)}{x^2-9} = \lim_{x\to -3} \left(x^2+9\right) = (-3)^2 + 9 = 9 + 9 = 18 \quad ■$$

3. Using the practical rules for computing limits, find $\displaystyle\lim_{x\to 6} |x-6| \cdot \cos\left(\frac{1}{x-6}\right)$. *[1.5]*

Solution. It's easy to see that $|x-6| \to 0$ as $x \to 6$. Unfortunately, $\cos\left(\frac{1}{x-6}\right)$ is undefined at $x = 6$ and oscillates infinitely often between -1 and 1 as $x \to 6$. We can take advantage of the fact that the oscillation is bounded in scale to apply the Squeeze Theorem:

Since $-1 \le \cos(t) \le 1$ for all real numbers t, it follows that for all $x \ne 6$ we have

$$-|x-6| = |x-6| \cdot (-1) \le |x-6| \cdot \cos\left(\frac{1}{x-6}\right) \le |x-6| \cdot 1 = |x-6| \,.$$

Since $\lim_{x\to 6}\left(-|x-6|\right) = -|6-6| = -|0| = 0$ and $\lim_{x\to 6}|x-6| = |6-6| = |0| = 0$, it follows by the Squeeze Theorem that $\displaystyle\lim_{x\to 6} |x-6| \cdot \cos\left(\frac{1}{x-6}\right) = 0$ as well. ■