Examples* of Substitution III

* ... just three more.

Warm-up: $$\int_0^1 \frac{e^z+1}{e^z+z}\,dz$$

$$\frac{d}{dz}(e^z+z) = e^z+1$$

so $w = e^z + z$

$dw = (e^z+1)\,dz$

z	w
0	1
1	$e+1$

$$= \int_1^{e+1} \frac{1}{w}\,dw$$

$$= \ln(w)\Big|_1^{e+1}$$

$$= \ln(e+1) - \ln(1)$$

$$= \ln(e+1) - 0$$

$$= \ln(e+1)$$

$$\int x^3 (x^2+1)^{1/2}\, dx = \int x^3 \sqrt{x^2+1}\, dx$$

$u = x^2$

$$= \int x^2 \cdot x \sqrt{x^2+1}\, dx$$

$du = 2x\,dx$

$\frac{1}{2} du = x\,dx$

$$= \int u \sqrt{u+1}\, \tfrac{1}{2} du$$

$$= \frac{1}{2} \int u \sqrt{u+1}\, du$$

$w = u+1 \qquad u = w-1$

$dw = du$

$$= \frac{1}{2} \int (w-1) \sqrt{w}\, dw$$

$$= \frac{1}{2} \int (w\sqrt{w} - \sqrt{w})\, dw$$

$$= \frac{1}{2} \int (w^{3/2} - w^{1/2})\, dw$$

$$= \frac{1}{2} \left(\frac{w^{5/2}}{5/2} - \frac{w^{3/2}}{3/2} \right) + C$$

$$= \frac{1}{2} \left(\frac{2}{5} w^{5/2} - \frac{2}{3} w^{3/2} \right) + C$$

$$= \frac{1}{5} w^{5/2} - \frac{1}{3} w^{3/2} + C$$

$$= \frac{1}{5} (u+1)^{5/2} - \frac{1}{3} (u+1)^{3/2} + C$$

$$= \frac{1}{5} (x^2+1)^{5/2} + \frac{1}{3} (x^2+1)^{3/2} + C$$

③

$$\int x^3 (x^2+1)^{1/2}\,dx = \int x^2 (x^2+1)^{1/2}\,x\,dx$$

$w = x^2+1 \quad x^2 = w-1$

$dw = 2x\,dx$

$\frac{1}{2}dw = x\,dx$

$$= \int (w-1)\, w^{1/2} \cdot \frac{1}{2}\,dw$$

$$= \frac{1}{2}\int \left(w^{3/2} - w^{1/2}\right) dw$$

$$= \frac{1}{2}\left(\frac{w^{5/2}}{5/2} - \frac{w^{3/2}}{3/2}\right) + C$$

$$= \frac{1}{5} w^{5/2} - \frac{1}{3} w^{3/2} + C$$

$$= \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C$$

$$\int_{-2}^{2} (x+3)\sqrt{4-x^2}\,dx$$

$$= \int_{-2}^{2} x\sqrt{4-x^2}\,dx \quad + \quad \int_{-2}^{2} 3\sqrt{4-x^2}\,dx$$

$u = 4-x^2$

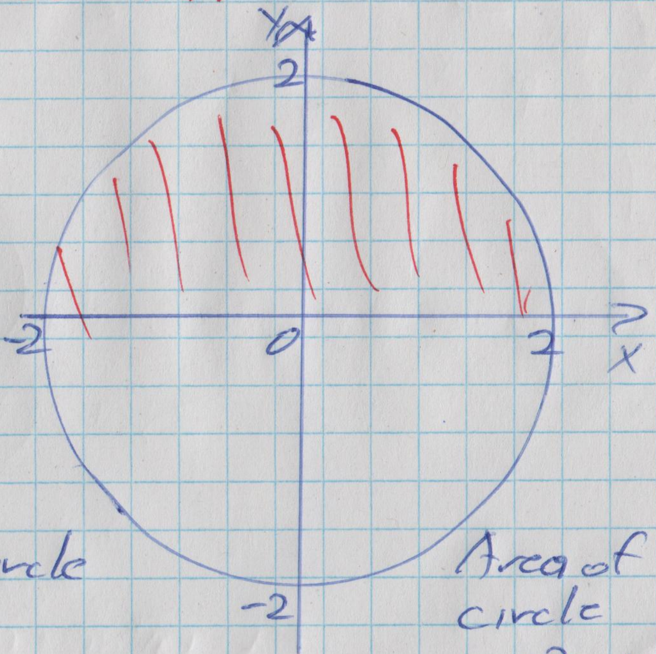
$du = -2x\,dx$

$\left(-\frac{1}{2}\right)du = x\,dx$

x	u
-2	0
2	0

upper half of the circle

Note: if $y = \sqrt{4-x^2}$, then $y^2 = 4-x^2$ and so $x^2+y^2 = 4$ ... which is the equation of a circle of radius 2 with centre at $(0,0)$



$$= \int_{0}^{0} \sqrt{u}\left(\frac{-1}{2}\right)du \quad + 3\int_{-2}^{2} \sqrt{4-x^2}\,dx$$

$= \quad 0 \quad + \quad 3 \cdot$ area of half of the circle

Area of circle $= \pi \cdot 2^2 = 4\pi$

$$= 3 \cdot \frac{4\pi}{2} = 6\pi$$

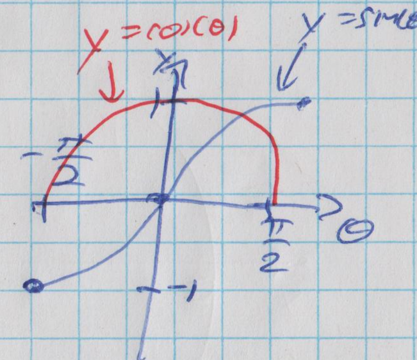
The integral $\int_{-2}^{2} \sqrt{4-x^2}\,dx$ <u>can</u> be solved by a substitution, but it needs a more sophisticated one: Substitute for x, by putting $x = 2\sin(\theta)$.

Then $dx = 2\cos(\theta)\,d\theta$, since $\frac{dx}{d\theta} = \frac{d}{d\theta} 2\sin(\theta) = 2\cos(\theta)$.

Changing the limits:

x	θ
-2	$-\frac{\pi}{2}$
2	$+\frac{\pi}{2}$

$x = 2\sin(\theta)$
$\Rightarrow -2 = 2\sin(\theta)$
$\Rightarrow \sin(\theta) = -1$
$\Rightarrow$



$x = 2\sin(\theta)$
$\Rightarrow 2 = 2\sin(\theta)$
$\Rightarrow \sin(\theta) = 1$
$\Rightarrow \theta = \frac{\pi}{2}$

Thus $\int_{-2}^{2} \sqrt{4-x^2}\,dx$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{4-4\sin^2(\theta)} \cdot 2\cos(\theta)\,d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2} \sqrt{4(1-\sin^2(\theta))}\,\cos(\theta)\,d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2} 2\sqrt{1-\sin^2(\theta)}\,\cos(\theta)\,d\theta$$

$\cos^2(\theta) + \sin^2(\theta) = 1$
so $\cos^2(\theta) = 1 - \sin^2(\theta)$

$$= 4\int_{-\pi/2}^{\pi/2} \sqrt{\cos^2(\theta)}\,\cos(\theta)\,d\theta = 4\int_{-\pi/2}^{\pi/2} \cos(\theta)\cos(\theta)\,d\theta$$

$$= 4\int_{-\pi/2}^{\pi/2} \cos^2(\theta)\,d\theta$$

$$\cos(2\theta) = 2\cos^2(\theta) - 1$$

$$\Rightarrow \cos^2(\theta) = \frac{1}{2}\left[1+\cos(2\theta)\right]$$

$$= 4\int_{-\pi/2}^{\pi/2} \frac{1}{2}\left[1+\cos(2\theta)\right]d\theta = 2\int_{-\pi/2}^{\pi/2}\left(1+\cos(2\theta)\right)d\theta$$

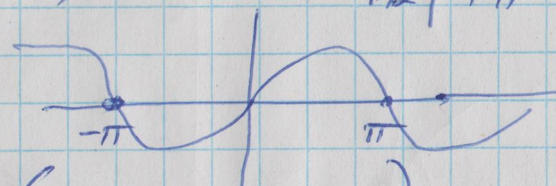
$u = 2\theta$

$du = 2\,d\theta$

$d\theta = \frac{1}{2}du$

θ	u
$-\pi/2$	$-\pi$
$+\pi/2$	$+\pi$

$$= 2\int_{-\pi}^{\pi}\left(1+\cos(u)\right)\cdot\frac{1}{2}\,du = \frac{\cancel{2}}{\cancel{2}}\int_{-\pi}^{\pi}\left(1+\cos(u)\right)du$$



$$= \left(u + \sin(u)\right)\Big|_{-\pi}^{\pi} = \left(\pi + \sin(\pi)\right) - \left(-\pi - \sin(-\pi)\right)$$

$$= (\pi + 0) - (-\pi - 0) = \pi - (-\pi) = 2\pi \ \checkmark$$

This kind of "trigonometric substitution" will be much revisited in MATH 1120H