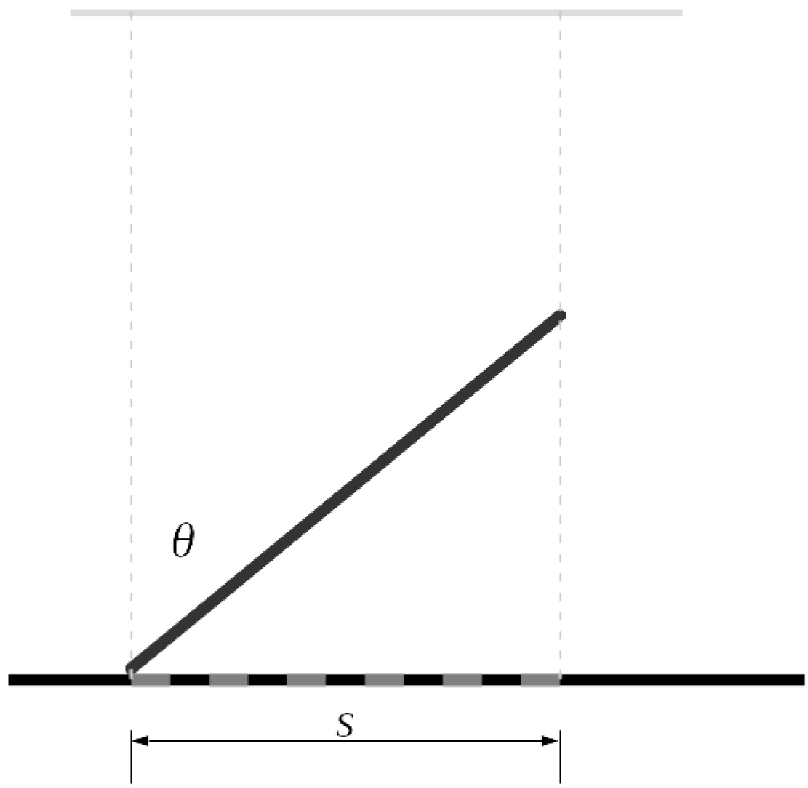
Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Fall 2020

Solution to Quiz #8
Tuesday, 17 November .

1. A lever 30 cm long is in the middle of a horizontal table, with it's hinge built into the surface of the table and with large ceiling lighting panel directly above the lever. The lever is pulled down from a vertical position to a horizontal position in such a way that the angle it makes with the vertical changes at a constant rate of $\pi/6$ rad/min. The lever casts a shadow on the surface of the table in the light from the panel. How is the length of this shadow changing at the instant that the angle the lever makes with the vertical is $\pi/3$ rad? *[5]*

SOLUTION. Here is a sketch of the setup:



We are given that the lever is 30 cm long, that $\frac{d\theta}{dt} = \frac{\pi}{6}$ rad/min (note that as the lever is pulled down, the angle it makes with the vertical is increasing, so $\frac{d\theta}{dt}$ must be positive), and that at the instant we're interested in, $\theta = \frac{\pi}{3}$ rad. Let s denote the length of the shadow that the lever casts on the table.

It is obvious from the sketch that the shadow is a short side of a right-angle triangle of which the lever is the hypotenuse. It is also clear that the angle between the lever and the surface of the table, and hence the shadow, is $\frac{\pi}{2} - \theta$ *rad*. It follows that

$$\cos\left(\frac{\pi}{2} - \theta\right) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{s}{30}\,,$$

so

$$s = 30\cos\left(\frac{\pi}{2} - \theta\right)\,.$$

If we were suitably gung-ho about the trig functions, we could rewrite $\cos\left(\frac{\pi}{2} - \theta\right)$ in terms of $\sin(\theta)$, but this is just enough trouble that we may as well proceed to taking derivatives without that simplification first.

$$\begin{aligned}\frac{ds}{dt} &= \left[\frac{d}{d\theta}30\cos\left(\frac{\pi}{2} - \theta\right)\right]\cdot\frac{d\theta}{dt} = -30\sin\left(\frac{\pi}{2} - \theta\right)\cdot\left[\frac{d}{d\theta}\left(\frac{\pi}{2} - \theta\right)\right]\cdot\frac{\pi}{6}\\ &= -30\sin\left(\frac{\pi}{2} - \theta\right)\cdot(-1)\cdot\frac{\pi}{6} = 5\pi\sin\left(\frac{\pi}{2} - \theta\right)\ cm/min\end{aligned}$$

When $\theta = \frac{\pi}{3}$, this gives:

$$\left.\frac{ds}{dt}\right|_{\theta=\pi/3} = 5\pi\sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = 5\pi\sin\left(\frac{\pi}{6}\right) = 5\pi\frac{1}{2} = \frac{5\pi}{2}\ cm/min$$

Thus, at the instant that $\theta = \frac{\pi}{3}$, the length of the shadow cast by the lever is increasing at a rate of $\frac{5\pi}{2} \approx 7.854\ cm/min$. ■