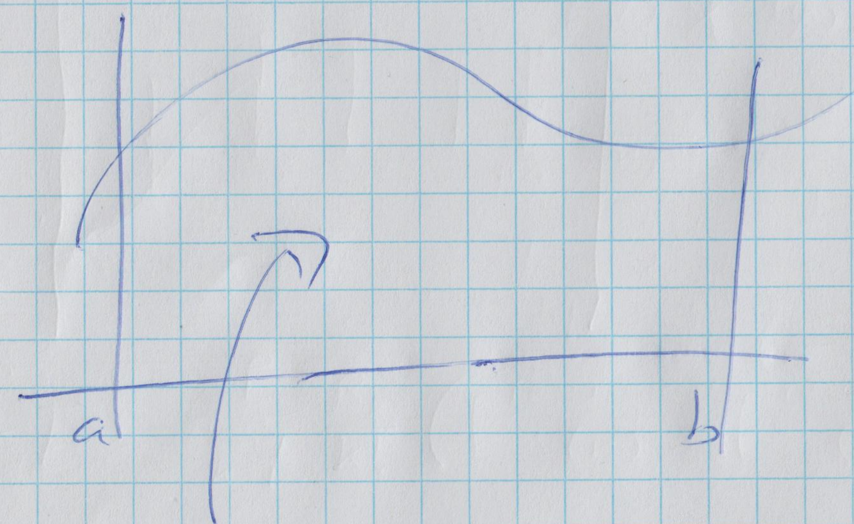
The Definite Integral – an informal look

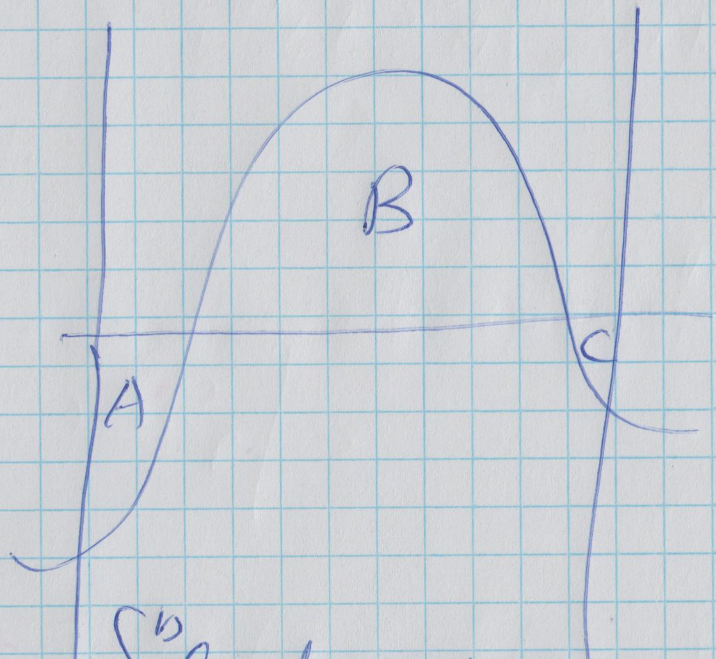
Where differentiation is basically finding slopes of tangent line, integration is finding (weighted) areas.

The definite integral $\int_a^b f(x)\,dx$ gives the area between $y = f(x)$ and the x-axis, where area above the x-axis is added and area below the x-axis is subtracted.

eg



$\int_a^b f(x)\,dx$



$$\int_a^b f(x)\,dx = -A + B - C$$

It's hard to define this precisely in a way that gives you all the intuitive properties this ought to have.

[See the handout <u>A Precise Definition of the Definite Integral</u>.

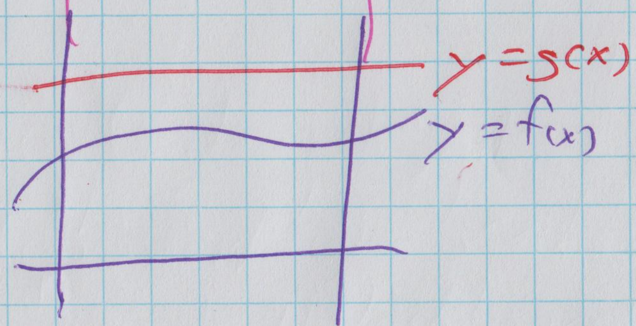
In the Supplementary Materials folder in the Course Content section on Blackboard, and on the archive page at euclid.trentu.ca/math/sb/1110H .]

We'll stick to dealing with this informally for the most part. [The textbook tries to slide the definition under the rug.]

The important thing for us will be the properties of the definite integral and the techniques used to compute them.

Properties of the definite integral:

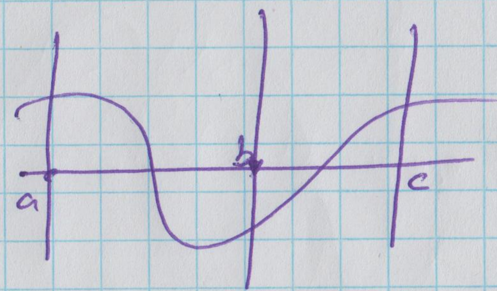
1° $\displaystyle\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$



2° $\displaystyle\int_a^b Cf(x)\,dx = C\int_a^b f(x)\,dx$

"The definite integral is <u>linear</u> in the functions input into it."

3° $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$



Consequences of this property:

a) $\displaystyle\int_a^a f(x)\,dx = 0$ b) $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

4° If $f(x) \geq 0$ on $[a,b]$,

then $\int_a^b f(x)\,dx \geq 0$.

5° $\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$

6° If $f(x)$ is continuous on $[a,b]$, and $\int_a^b |f(x)|\,dx = 0$,

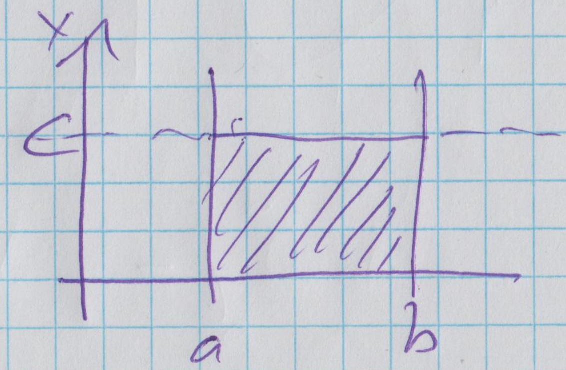
then $f(x) = 0$ for all $x \in [a,b]$.

A couple of special cases with easy geometry:

i) $\int_a^b c\,dx = c(b-a)$

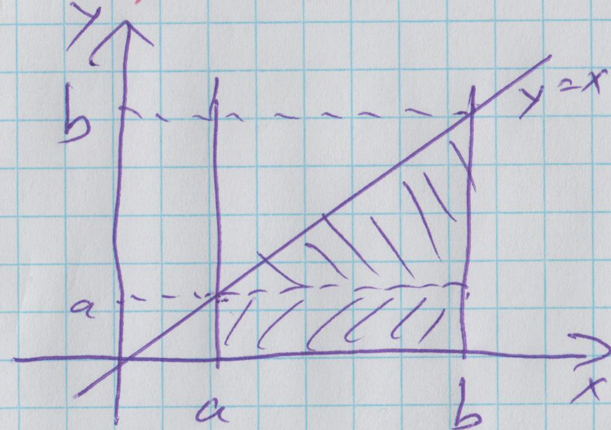
height width

(C a constant)



ii) $$\int_a^b x\,dx = \frac{b^2-a^2}{2}$$

$$= \underset{\text{area rectangle}}{\overset{\text{height}}{a}\overbrace{(b-a)}^{\text{width}}} + \underbrace{\frac{1}{2}\underset{\text{base}}{(b-a)}\underset{\text{height}}{(b-a)}}_{\text{area triangle}}$$



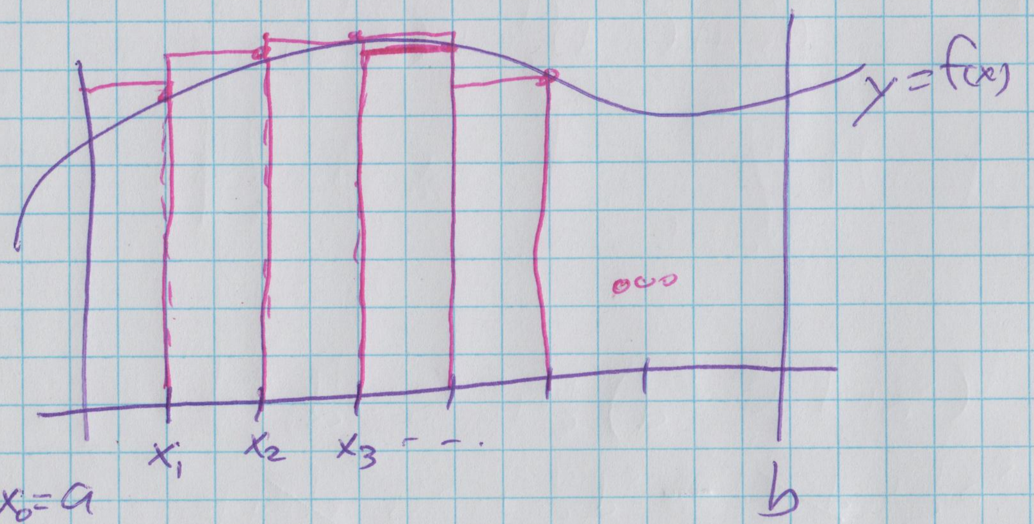
$$= ab - a^2 + \frac{1}{2}(b^2 - 2ab + a^2)$$

$$= \cancel{ab} - a^2 + \frac{1}{2}b^2 \cancel{- \frac{1}{2}2ab} + \frac{1}{2}a^2 \qquad = \frac{1}{2}b^2 - \frac{1}{2}a^2 = \frac{b^2-a^2}{2}$$

In most cases, the geometry is not so favourable, which is we have to difficult definitions for $\int_a^b f(x)dx$.

If $f(x)$ is continuous on $[a,b]$, then there are all sorts of limit formulas that are simplications of the real definition, that compute $\int_a^b f(x)dx$.

Perhaps the simplest is the Right-Hand Rule:



Each of the n equal subintervals of $[a,b]$ has width $\frac{b-a}{n}$, so the subintervals are

$$[x_0,x_1] = \left[a, a+\frac{b-a}{n}\right],$$
$$[x_1,x_2] = \left[a+\frac{b-a}{n}, a+\frac{2(b-a)}{n}\right]$$
$$[x_2,x_3] = \left[a+2\frac{b-a}{n}, a+3\frac{b-a}{n}\right]$$
$$\vdots$$
$$[x_{n-1},x_n] = \left[a+(n-1)\frac{b-a}{n}, \underbrace{a+n\frac{b-a}{n}}_{b}\right]$$

Divide $[a,b]$ into n equal pieces and use the right-hand endpoint and f(x) to compute the height of a rectangle whose base is the subinterval.

These areas of these rectangles between them approximate $\int_a^b f(x)dx$

So $x_k = a+k\frac{b-a}{n}$ for $k=0,1,\dots,n$,

and then the sum of the areas of the rectangles is:

1st rect. 2nd rect nth rect

$$\underbrace{\frac{b-a}{n}}_{\text{width}}\cdot \underbrace{f\left(a+1\cdot\frac{b-a}{n}\right)}_{\text{height}} + \frac{b-a}{n}\cdot f\left(a+2\frac{b-a}{n}\right) + \cdots + \frac{b-a}{n}\cdot f(\underset{\substack{'' \\ b+n\frac{b-a}{n}}}{b})$$

$$= \sum_{i=1}^{n} \underbrace{\frac{b-a}{n}\cdot f\left(a+i\,\frac{b-a}{n}\right)}_{\text{area of } i^{th} \text{ rectangle}}$$

"Sum"

This is the Right-Hand Rule approximation to $\int_a^b f(x)\,dx$ where $[a,b]$ is partitioned into n equal subintervals.

"n^{th} Right-Hand Rule sum for $\int_a^b f(x)\,dx$"

__Fact:__ $\lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{b-a}{n}\cdot f\left(a+i\,\frac{b-a}{n}\right)\right]$

$$= \int_a^b f(x)\,dx$$

provided that $f(x)$ is defined on $[a,b]$ and has at most finitely many removable or jump discontinuities in $[a,b]$.

Nothing too special about using right endpoints... There are Left-Hand and Midpoint Rules as well. Also, more sophisticated rules:

Trapezoid Rules

Simpson's Rule, etc.

We can't quite use this as a definition of $\int_a^b f(x)dx$.

For example: Hard to get
$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$
using this a def'n. since a partition of $[a,b]$ into equal pieces cannot in some cases be extended into a partition of ~~$[a,b]$~~ $[b,c]$ into equal pieces as well.

> The general (hard!) def'n of $\int_a^b f(x)dx$ uses arbitrary partitions of $[a,b]$, not necessarily equal in width.
>
> [But we're avoiding this, for now.]

Eg $[0,1]$ $[1,\sqrt{2}]$

chop into equal pieces

chop into equal pieces

they can't match

since $1-0=1$ is rational

but $\sqrt{2}-1$ is irrational because $\sqrt{2}$ is irrational

so $\dfrac{1-0}{n} = \dfrac{\sqrt{2}-1}{m}$ is impossible, since then $\sqrt{2} \neq m\left(\frac{1}{n}+\frac{1}{m}\right)$.