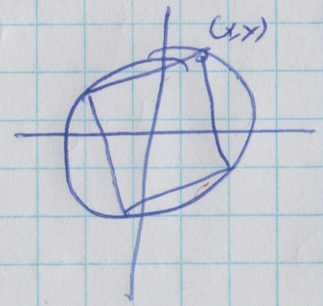
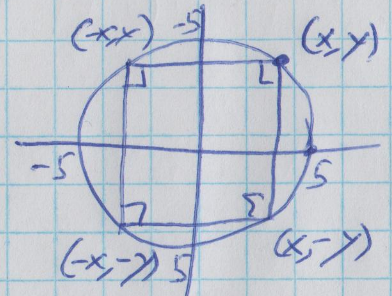
Optimization (i.e. applied max/min) by example

Problem: Find the maximum area of a rectangle whose corners are all on the circle $x^2+y^2=25$.

Simplification:



is easier with the sides parallel to the axes (makes it easier to find the lengths of the sides to compute the area).



Then the area of this rectangle is width·height $= (x-(-x))(y-(-y))$

$$= 2x \cdot 2y = 4xy.$$

Refined problem: We want to maximize $4xy$ subject to the requirements

$$x^2+y^2=25, \quad 0 \le x, \quad 0 \le y,$$

$$x \le 5, \quad y \le 5.$$

Put it all in terms of one variable, say x.

$y = \sqrt{25-x^2}$ (positive root since $y \ge 0$).

So we really want to maximize

$$A(x) = 4xy = 4x\sqrt{25-x^2}, \quad \text{where } 0 \le x \le 5$$

Note that $A(0) = 0 = A(5)$.

Look for critical points in $[0,5]$:

$$A'(x) = \frac{d}{dx}4x\sqrt{25-x^2} = 4\cdot 1\cdot\sqrt{25-x^2} + 4x\frac{1}{\cancel{2}\sqrt{25-x^2}}\cdot(-\cancel{2}x)$$

$$= 4\left(\sqrt{25-x^2} - \frac{\cancel{2}x^2}{\sqrt{25-x^2}}\right) = 0$$

$$\Leftrightarrow 4\left(\sqrt{25-x^2} - \frac{\cancel{4}x^2}{\sqrt{25-x^2}}\right)\sqrt{25-x^2} = 0\cdot\sqrt{25-x^2} = 0$$

$$= 4\left((25-x^2) - \cancel{4}x^2\right) = 4(25-2x^2)$$

$$\Leftrightarrow 25-2x^2 = 0 \Leftrightarrow x^2 = \frac{25}{2} \Leftrightarrow x = \pm\frac{5}{\sqrt{2}}$$

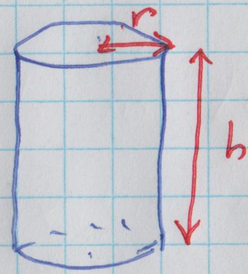
Since $x \ge 0$, we only have to worry about $x = +\frac{5}{\sqrt{2}}$.

$$A\left(\tfrac{5}{\sqrt{2}}\right) = 4\cdot\frac{5}{\sqrt{2}}\sqrt{25-\left(\tfrac{5}{\sqrt{2}}\right)^2} = 10\sqrt{2}\cdot\sqrt{25-\tfrac{25}{2}} = 10\cancel{\sqrt{2}}\cdot\frac{\sqrt{25}}{\cancel{\sqrt{2}}} = 50.$$

Since $50 > 0 = A(0) = A(5)$, the maximum area of a rectangle with corners on $x^2+y^2=25$ is 50.

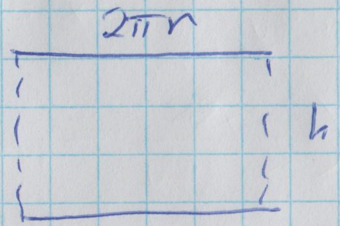
Problem: What is the minimum possible surface area of a cylindrical can with volume 1 L?

Simplification / refinement:



$$V = \pi r^2 h = 1$$

$$SA = \underbrace{2\pi r^2}_{\text{ends}} + \underbrace{2\pi r h}_{\text{sides}}$$



Put in terms of one variable

solve $\pi r^2 h = 1$ for r or h.

We'll solve for h because it's a bit easier $h = \frac{1}{\pi r^2}$,

where

$0 \leq r < \infty$

So we want to minimize $SA(r) = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \cdot \frac{1}{\pi r^2}$

$$= 2\pi r^2 + \frac{2}{r}$$

on $(0, \infty)$.

Check the "endpoints":

$$\lim_{r\to 0^+} (\underbrace{2\pi r^2}_{\downarrow \atop 0^+} + \underbrace{\frac{2}{r}}_{\downarrow \atop +\infty}) = +\infty \qquad \text{no minimum here!}$$

$$\lim_{r\to \infty} (\underbrace{2\pi r^2}_{\downarrow \atop +\infty} + \underset{\downarrow \atop 0^+}{\frac{2}{r}}) = +\infty \qquad \text{——— " ———}$$

Check the critical points in $(0,\infty)$:

$$SA'(r) = \frac{d}{dr}\left(2\pi r^2 + \frac{2}{r}\right) = 2\pi \cdot 2r + 2\cdot\frac{-1}{r^2} = 4\pi r - \frac{2}{r^2}$$

$$= 0 \iff 4\pi r - \frac{2}{r^2} = 0 \iff 2\pi r - \frac{1}{r^2} = 0$$

$$\iff 2\pi r^3 - 1 = 0 \iff r^3 = \frac{1}{2\pi}$$

$$\iff r = \sqrt[3]{\frac{1}{2\pi}} = \left(\frac{1}{2\pi}\right)^{1/3} = (2\pi)^{-1/3} \quad > 0$$

$$SA\left((2\pi)^{-1/3}\right) = 2\pi\,(2\pi)^{-2/3} + \frac{2}{(2\pi)^{-1/3}} = (2\pi)^{1/3} + 2(2\pi)^{1/3}$$

$$= 3(2\pi)^{1/3} \text{ (in decimeters)}$$

$$\left(1\,(dm)^3 = 1\,L\right) \qquad = 30(2\pi)^{1/3} \text{ cm}$$

(& now use a calculator if you care about the actual number)