Maxima and Minima III

To find all the maxima and minima of $f(x)$ on an interval proceed as follows

1. Check for local maxima and minima by evaluating $f(x)$ at all points in the interval where $f'(x)$ is $=0$ or is undefined. (You can check if such a point is a local max or min or neither, by testing points on either side or by checking how the derivative changes from one side to the other.)

2. Check what happens at points in the interval at which $f(x)$ is undefined, ie take the limits of $f(x)$ from each side at such points.

3. Check what happens at endpoints:
   - evaluate $f(x)$ at any ~~point~~ endpoints that are in the interval
   - take the limit of $f(x)$ [from within the interval] at endpoints that are not in the interval.

4. Compare all of these values [some might be $\pm\infty$].

(a) If the largest or smallest value is a real number at a point in the interval at which $f(x)$ is defined and continuous, then that is the, respectively, absolute maximum or minimum value of $f(x)$ on the interval. [There may be more than one such maximum or minimum point.]

(b) If the largest or smallest value (but is real) is not a point of the interval that $f(x)$ is defined and continuous on, then it is a least upper or (respectively) greatest lower bound for $f(x)$ on the interval, but is not actually achieved by $f(x)$ on the interval. [so technically no abs max, resp. min]

(c) ~~If neither of the above occurs for~~
If the largest value generated was $+\infty$
resp. smallest value generated was $-\infty$,
then $f(x)$ has no abs. max or upper bound [resp.] no abs. min or lower bound on the interval.

$$f(x) = \frac{2x^2-x-6}{-x^2+3x-2} \quad \text{on } (-\infty, \infty)$$

1 $$f'(x) = \frac{d}{dx}\left(\frac{2x^2-x-6}{-x^2+3x-2}\right) = \frac{\left[\frac{d}{dx}(2x^2-x-6)\right](-x^2+3x-2) - (2x^2-x-6)\left[\frac{d}{dx}(-x^2+3x-2)\right]}{(-x^2+3x-2)^2}$$

$$= \frac{(4x-1)(-x^2+3x+2) - (2x^2-x-6)(-2x+3)}{(-x^2+3x-2)^2}$$

$$= \frac{(\cancel{-4x^3}+13x^2+5x-2) - (\cancel{-4x^3}+6x^2+9x-18)}{(-x^2+3x-2)^2}$$

$$= \frac{7x^2-4x+16}{(-x^2+3x-2)^2}$$

$$\begin{array}{r} 28 \\ \times 16 \\ \hline 168 \\ 280 \\ \hline 448 \end{array}$$

Next: figure out where $f'(x) = 0$ and where $f'(x)$ is not defined.

$$f'(x) = 0 \iff 7x^2-4x+16 = 0 \iff x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4\cdot 7\cdot 16}}{2\cdot 7}$$

$$= \frac{4 \pm \sqrt{16-448}}{14}$$

$$= \frac{4 \pm \sqrt{-432}}{14}$$

$\therefore$ there are no points where $f'(x) = 0$ since there are no real solutions

$f'(x)$ in undefined exactly when $(-x^2+3x-2)^2 = 0$

iff — " — $-x^2+3x-2 = 0$

— " — $x = \dfrac{-3 \pm \sqrt{3^2 - 4\cdot(-1)\cdot(-2)}}{2\cdot(-1)}$

$= \dfrac{-3 \pm \sqrt{9-8}}{-2} = \dfrac{-3 \pm 1}{-2}$

$= \dfrac{-4}{-2}$ or $\dfrac{-2}{-2}$

$= +2$, $= +1$

But we can't evaluate $f(x)$ at these points, because $f(x)$ has $-x^2+3x-2$ as its denominator, so it is undefined at these points.

So we have no critical points to test.

(2) Check points where $f(x)$ is undefined, $x=1$ & $x=2$.

$$f(x) = \frac{2x^2-x-6}{-x^2+3x-2} = \frac{(x-2)(2x+3)}{-(x-1)(x-2)} = \frac{2x+3}{-(x-1)} = \frac{2x+3}{1-x}$$

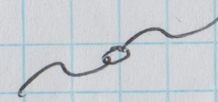
↑ except when $x=2$ where the original is undefined.

$$= \begin{cases} \dfrac{2x+3}{1-x} & x \neq 2 \\ \text{undef} & x = 2 \end{cases}$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{2x+3}{1-x} = \frac{2\cdot 2+3}{1-2} = \frac{7}{-1} = -7$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{2x+3}{1-x} = \frac{2\cdot 2+3}{1-2} = \frac{7}{-1} = -7$$

[This means we have a "removable discontinuity" at $x=2$


]

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{\overset{\to 2\cdot 1+3=5}{2x+3}}{\underset{\to 0^+}{1-x}} = +\infty$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{\overset{\to 5}{2x+3}}{\underset{\to 0^-}{1-x}} = -\infty$$

So we have a V.A. (on both sides) at $x=1$ & $f(x)$ goes in opposite directions from opposit sides.

(3) Check endpoints of $(-\infty, \infty)$

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{\overset{\to -\infty}{2x+3}}{\underset{\to +\infty}{1-x}} = \lim_{x\to -\infty} \frac{\frac{d}{dx}(2x+3)}{\frac{d}{dx}(1-x)} = \lim_{x\to -\infty} \frac{2}{-1} = -2$$

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \frac{\overset{\to +\infty}{2x+3}}{\underset{\to -\infty}{1-x}} = \lim_{x\to +\infty} \frac{\frac{d}{dx}(2x+3)}{\frac{d}{dx}(1-x)} = \lim_{x\to +\infty} \frac{2}{-1} = -2$$

So we have a HA of $y=-2$ ~~in~~ both both directions.

4 – No critical points ($f'(x) \neq 0$ for all x, and $f'(x)$ is undefined only at points where $f(x)$ is also undefined.)

– pts where $f(x)$ is undefined generated values of limits (from each direction at each point.) of $-\infty$, -7, $+\infty$

– "endpoints" generated limits of -2 in each case

No local maxima or minima (no critical points).

No abs. max or min (because $+\infty$ & $-\infty$ are on the list above)

(and also no upper or lower bounds)