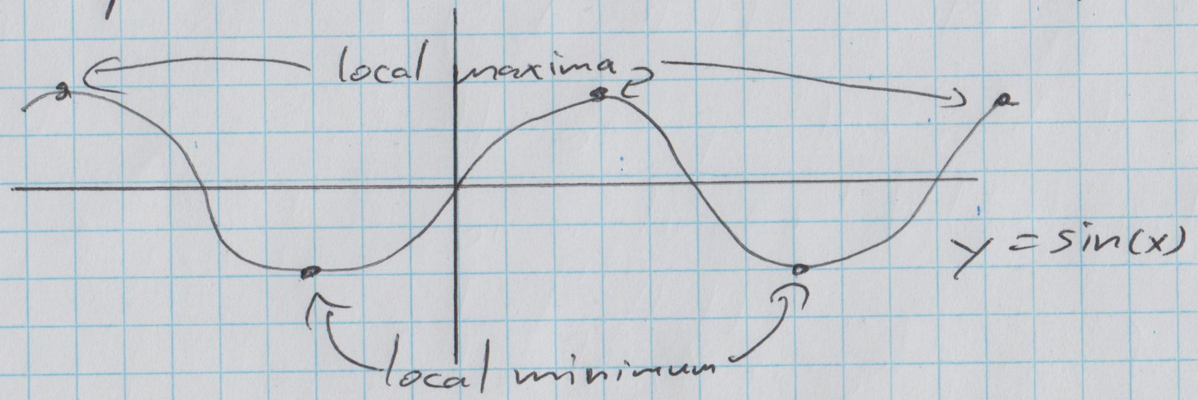
2020-10-09

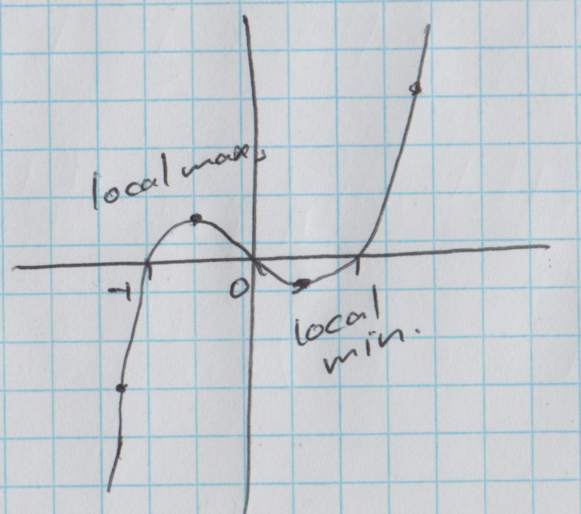
Maxima and Minima - Some basics

<u>Def'n:</u> $f(x)$ has a local minimum (resp. maximum) at a if for all points x near a $f(x) \geq f(a)$ (resp, $f(x) \leq f(a)$).

<u>eg</u>

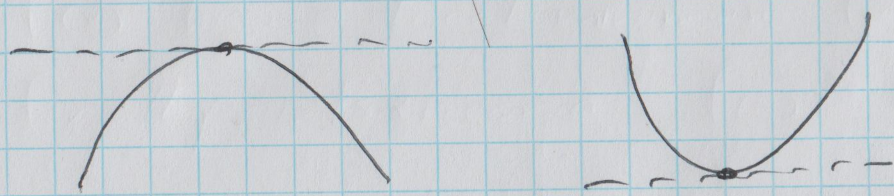


<u>eg</u> $y = x^3 - x = x(x-1)(x+1)$



Note that far enough way from a local min (resp. max) the function can get below the local min (resp. above the local max.)

At local min's & max's differentiable functions have horizontal tangents



ie the derivative is 0 at such points

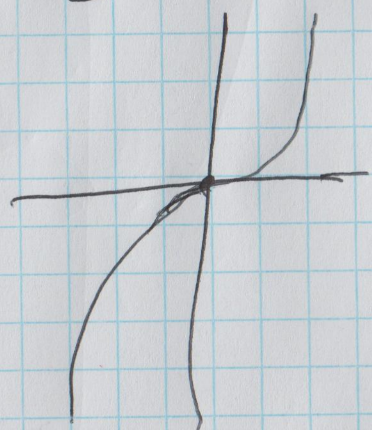
Fermat's Theorem

If $f(x)$ is differentiable near a point a (and at a itself) and it has a local maximum or minimum at a, then $f'(a) = 0$.

Pierre de Fermat, along with Descartes, invented the Cartesian coordinate system.

Note: It's possible for $f'(a) = 0$ at some $x = a$ without that function having a local min. or max at $x = a$.

eg $y = x^3$



$$\left.\frac{dy}{dx}\right|_{x=0} = \left.3x^2\right|_{x=0} = 3 \cdot 0^2 = 0$$

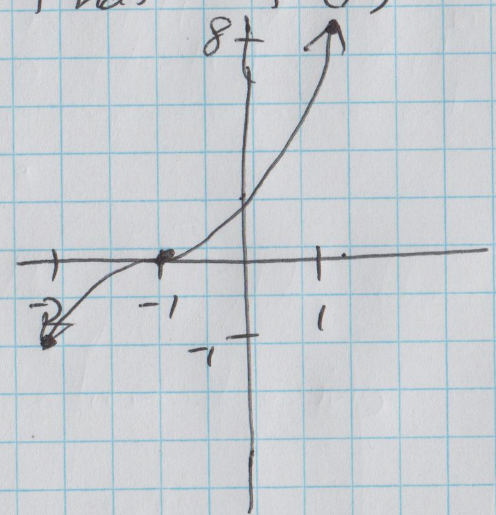
but $y = x^3$ is actually increasing everywhere. (So no local max or min anywhere.)

Ex What are the maximum and minimum values of $f(x) = x^3 + 3x^2 + 3x + 1$ on the interval $[-2, 1]$?

endpoints included

$$f'(x) = 3x^2 + 6x + 3 = 3(x^2+2x+1)$$
$$= 3(x+1)^2 = 0 \quad \text{exactly when} \quad x=-1$$

Note that $f'(x) = 3(x+1)^2 \geq 0$ for all x.



So the function $f(x)$ is increasing for all x except at $x=-1$.

So this "critical point" is neither a local max nor a local min.

Note that minimum value of $f(x)$ on $[-2,1]$ occurs at -2 ($f(-2) = -1$) and the maximum value of $f(x)$ on $[-2,1]$
min value

occurs at 1 ($f(1) = 8$).
max value

5. Find the absolute maximum and minimum of $f(x) = x\ln(x)$.

This is defined for $x > 0$, ie on the interval $(0, \infty)$, because $\ln(x)$ is only defined there.

Look for critical points: $f'(x) = \frac{d}{dx}\left(x\ln(x)\right) = \left(\frac{dx}{dx}\right)\cdot\ln(x) + x\cdot\left(\frac{d}{dx}\ln(x)\right)$

$= 1\cdot\ln(x) + x\cdot\frac{1}{x} = \ln(x) + 1$

So $f'(x) = 0$ exactly when $\ln(x) + 1 = 0$

$\iff \ln(x) = -1$

$\iff x = e^{\ln(x)} = e^{-1} = \frac{1}{e}$

(This is the only critical point.)

Is it a local max or min or neither?

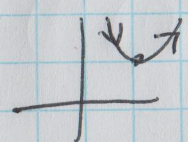
$f'(x) = \ln(x) + 1 > 0$ exactly when $\ln(x) > -1$

$\iff x > \frac{1}{e}$

< 0 exactly when $\ln(x) < -1$

$\iff x < \frac{1}{e}$

x	$< \frac{1}{e}$	$\frac{1}{e}$	$> \frac{1}{e}$
$f'(x)$	$-$	0	$+$
$f(x)$	$\downarrow$	local min	$\uparrow$



Be cause $f(x)$ (and $f'(x)$) are defined and continuous and differentiable on $(0,\infty)$ and there are no other critical points $f(x)$ has an absolute minimum at $\frac{1}{e}$

$$f\left(\tfrac{1}{e}\right) = \tfrac{1}{e}\ln\left(\tfrac{1}{e}\right) = \tfrac{1}{e}\ln(e^{-1}) = \tfrac{-1}{e}\ln(e) = -\tfrac{1}{e}\cdot 1 = \tfrac{-1}{e}$$

The "endpoints" $x=0$ and $x=\infty$ of the interval are not included in the interval, so we check them by taking limits.

l'Hôpital's Rule

$$\lim_{x\to 0^+} \underset{\substack{\downarrow\\0}}{x}\underset{\substack{\downarrow\\-\infty}}{\ln(x)} = \lim_{x\to 0^+} \frac{\ln(x) \to -\infty}{\frac{1}{x} \to +\infty} = \lim_{x\to 0^+} \frac{\frac{d}{dx}\ln(x)}{\frac{d}{dx}\left(\frac{1}{x}\right)} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}}$$

$$= \lim_{x\to 0^+} \frac{1}{\cancel{x}}\cdot\frac{x^{\cancel{2}}}{-1} = \lim_{x\to 0^+}(-x) = 0^-$$

$$\lim_{x\to+\infty} \underset{\substack{\downarrow\\+\infty}}{x}\underset{\substack{\downarrow\\+\infty}}{\ln(x)} = +\infty$$

Graph