Limits, or making "gets close to" precise.

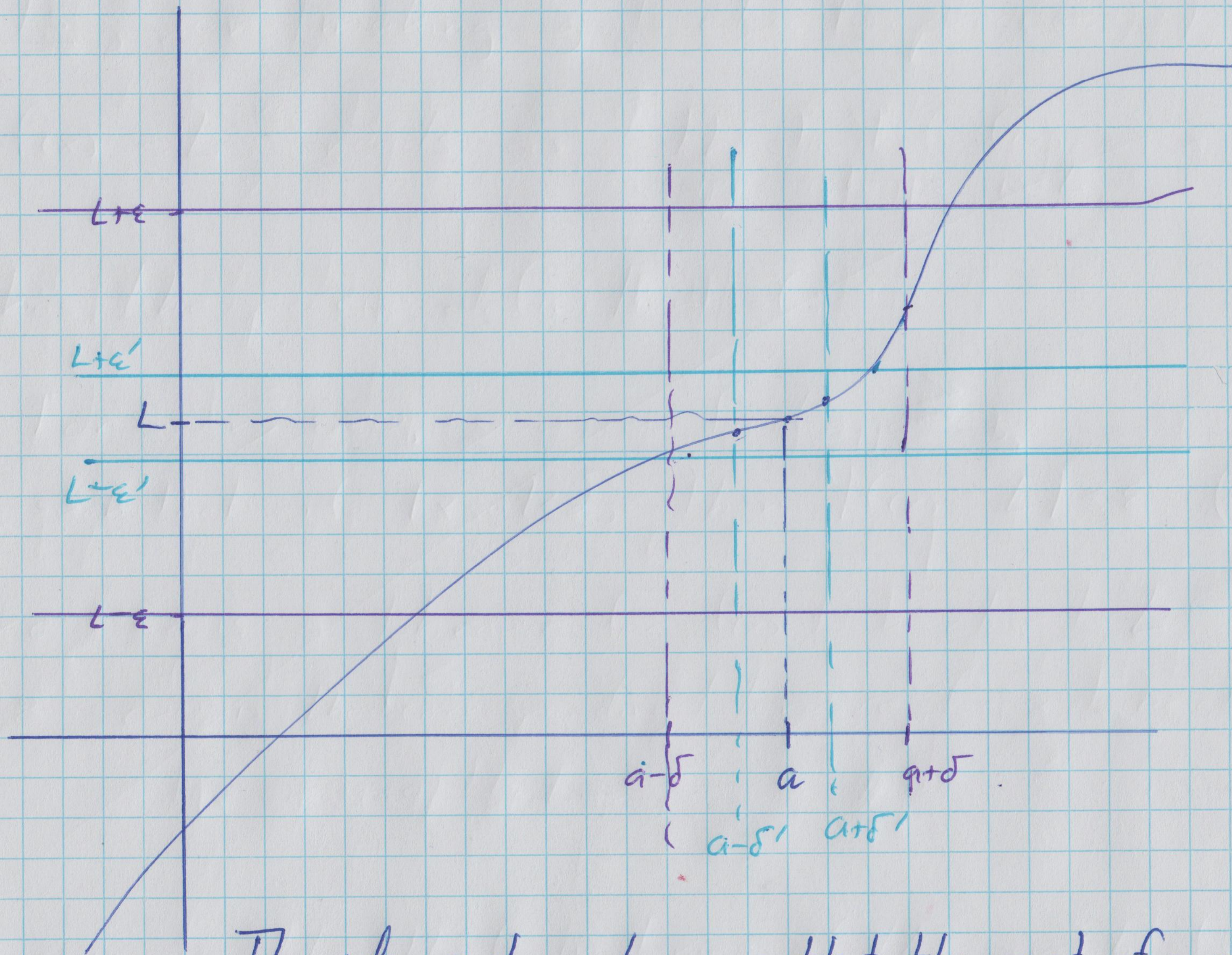
"$\lim_{x \to a} f(x) = L$" reads as "the limit of $f(x)$ as x approaches a is L" $\begin{pmatrix} a, L \\ \in \mathbb{R} \end{pmatrix}$

This has the precise definition of:

For every $\varepsilon > 0$, you can find a $\delta > 0$,
such that if $|x-a| < \delta$, then $|f(x) - L| < \varepsilon$ (for all x).

Here $\varepsilon > 0$ is a tolerance for $f(x)$ to be close to L
& $\delta > 0$ is the corresponding tolerance for x to be close to a
to ensure that.

This can be thought of in terms of the graph of $f(x)$.



Note that δ may depend on ε. (But can't be depending on the x's it controls.)

The idea is to make sure that the part of $y = f(x)$ for $a-\delta < x < a+\delta$ is actually also in the strip

$$L-\varepsilon < \underset{f(x)}{\underset{\shortparallel}{y}} < L+\varepsilon .$$

Verify that

eg $\lim_{x \to 1} (3x-2) = 1$ (Note that most functions are continuous most of the time, so $\lim_{x \to a} f(x) = f(a)$.)

That is, if we are given any $\varepsilon > 0$, we have to show there is a $\delta > 0$, such that if $|x-1| < \delta$, then $|(3x-2)-1| < \varepsilon$.

We'll reverse engineer the δ from the desired conclusion:

$$|(3x-2)-1| < \varepsilon$$
$$\Leftrightarrow |3x-3| < \varepsilon$$
$$\Leftrightarrow |3| \cdot |x-1| < \varepsilon$$
$$\Leftrightarrow 3|x-1| < \varepsilon$$
$$\Leftrightarrow |x-1| < \frac{\varepsilon}{3}$$

So $\delta = \frac{\varepsilon}{3}$ ought to work!

$\delta = \frac{\varepsilon}{3}$ does work:

if $|x-1| < \frac{\varepsilon}{3}$, then

$$3|x-1| < \varepsilon$$
$$\|$$
$$|3x-3|$$
$$\|$$
$$|(3x-2)-1| \quad \text{is } |(3x-2)-1| < \varepsilon$$

$\therefore \lim_{x \to 1} (3x-2) = 1.$

Note that the verification just reversed the engineering process.

eg Verify that $\lim_{x \to 2} (x^2 - 2x + 1) = 1$.

We need to check that for any $\varepsilon > 0$, we can find a corresponding $\delta > 0$ that ensures that

if $|x-2| < \delta$, then $|(x^2 - 2x + 1) - 1| < \varepsilon$.

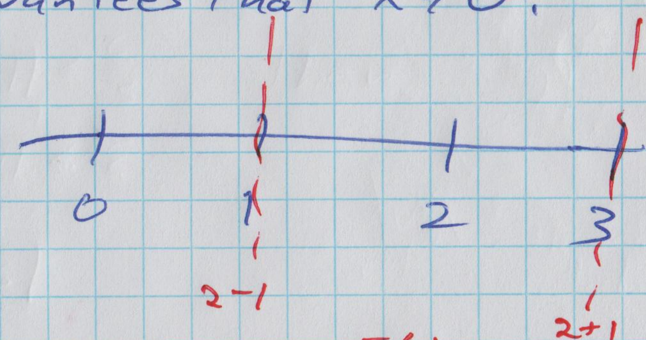
Try to reverse-engineer the δ:

$$|(x^2 - 2x + 1) - 1| < \varepsilon$$
$$\iff |x^2 - 2x| < \varepsilon$$
$$\iff |x(x-2)| < \varepsilon$$
$$\iff |x| \cdot |x-2| < \varepsilon$$
$$\overset{?}{\iff} |x-2| < \frac{\varepsilon}{|x|}$$

Two problems with the last step:

1. If $x = 0$, you'd be dividing by 0.
2. δ can't depend on x.

$\Rightarrow$ Work around this by picking only $\delta > 0$ such that $|x-2| < \delta$ guarantees that $x \neq 0$.



So use $\delta \leq 1$.

Then $|x-2| < \delta \leq 1$

$\Rightarrow |x-2| < 1 \Rightarrow -1 < x-2 < 1$

$\Rightarrow 1 < x < 3$

So if we accept only $\delta \leq 1$, we are guaranteed to have $1 < x < 3$. This means that

1) $x > 0$, so $x \neq 0$ (and $|x| = x$.)
2) $\frac{1}{1} > \frac{1}{x} > \frac{1}{3}$ ie $1 > \frac{1}{x} > \frac{1}{3}$

Since $\frac{1}{3} < \frac{1}{x} < 1$, we have $\frac{\varepsilon}{3} < \frac{\varepsilon}{x} = \frac{\varepsilon}{|x|} < \varepsilon \cdot 1 = \varepsilon$
(and $\varepsilon > 0$)

Thus if $\delta \leq 1$, and $|x-2| < 1$ and $|x-2| < \frac{\varepsilon}{3}$,

Then if also $|x-2| < \frac{\varepsilon}{3} < \frac{\varepsilon}{|x|}$ so $|x| \cdot |x-2| < \varepsilon$

$$|x(x-2)| = |x^2-2x| = |(x^2-2x+1)-1|$$

ie $|(x^2-2x+1)-1| < \varepsilon$.

Thus $\delta =$ minimum of 1 and $\frac{\varepsilon}{3}$
$= \min\left(1, \frac{\varepsilon}{3}\right)$

does the job.

$\therefore$ Since we can do this for any $\varepsilon > 0$, we have $\lim\limits_{x \to 2} (x^2 - 2x + 1) = 1$.

Imagine how much harder this likely gets when the powers of x go up... (Nevermind more difficult functions!)

So this definition is mainly used theoretically:

ie we prove various rules for manipulating limits using this definition and use to actually compute limits instead of using the definition directly.

The basic idea behind the ε-δ definition of limits is also used to do things analysing how errors propagate in a calculation.

Next time: look at the limit rules (or laws) & variations on the definition.