Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Fall 2019

Solutions to Assignment #2
Games With Limits

The usual $\varepsilon - \delta$ definition of limits,

> DEFINITION. $\lim\limits_{x \to a} f(x) = L$ exactly when for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any x with $|x - a| < \delta$ we are guaranteed to have $|f(x) - L| < \varepsilon$ as well.

is pretty hard to wrap your head around the first time or three for most people. Here is less common definition, equivalent to the one above, which recasts the confusing logical structure of the above definition in terms of a game:

> ALTERNATE DEFINITION. The *limit game* for $f(x)$ at $x = a$ with target L is a three-move game played between two players A and B as follows:
>
> 1. A moves first, picking a small number $\varepsilon > 0$.
> 2. B moves second, picking another small number $\delta > 0$.
> 3. A moves third, picking an x that is within δ of a, *i.e.* $a - \delta < x < a + \delta$.
>
> To determine the winner, we evaluate $f(x)$. If it is within ε of the target L, *i.e.* $L - \varepsilon < f(x) < L + \varepsilon$, then player B wins; if not, then player A wins.
>
> With this idea in hand, $\lim\limits_{x \to a} f(x) = L$ means that player B has a winning strategy in the limit game for $f(x)$ at $x = a$ with target L; that is, if B plays it right, B will win no matter what A tries to do. (Within the rules ... :-) Conversely, $\lim\limits_{x \to a} f(x) \neq L$ means that player A is the one with a winning strategy in the limit game for $f(x)$ at $x = a$ with target L.

The game definition of limits isn't really better or worse that the usual $\varepsilon-\delta$ definition, but each is easier for some people to understand, and the exercise in trying it both ways usually helps in understanding what is really going on here.

1. Use one of these two definitions of limit to verify that $\lim\limits_{x \to 1} (-2x + 1) = -1$. *[2]*

SOLUTION THE FIRST. *(Game definition)* We'll use the game version of the definition to verify that $\lim\limits_{x \to 1} (-2x + 1) = -1$. This means that we need to find a winning strategy for player B; that is, a way to choose a $\delta > 0$ in response to player A's playing an $\varepsilon > 0$ that ensures that whatever x with $|x - 1| < \delta$ player A may try on the last move, player B wins. As usual with these problems, we will try to reverse-engineer a suitable δ from the winning condition for player B, namely $|(-2x + 1) - (-1)| < \varepsilon$.

$$\begin{aligned} |(-2x+1) - (-1)| < \varepsilon \quad &\iff \quad |-2x + 1 + 1| < \varepsilon \quad \iff \quad |-2x + 2| < \varepsilon \\ &\iff \quad |-2(x-1)| < \varepsilon \quad \iff \quad |-2| \cdot |x - 1| < \varepsilon \\ &\iff \quad 2\,|x - 1| < \varepsilon \quad \iff \quad |x - 1| < \frac{\varepsilon}{2} \end{aligned}$$

Player B's strategy will thus be to play $\delta = \dfrac{\varepsilon}{2}$ in response to whatever $\varepsilon > 0$ player A plays on the first move.

Why is this a winning strategy? Consider how the limit game plays out with this strategy for B:

1. A plays their choice of an $\varepsilon > 0$.
2. Following the strategy, B responds with $\delta = \dfrac{\varepsilon}{2}$.
3. A plays their choice of an x with $|x-1| < \delta = \dfrac{\varepsilon}{2}$.

Who wins? Looking at the reverse-engineering of $\delta = \dfrac{\varepsilon}{2}$ above, we see that every step is reversible. Since $|x-1| < \delta = \frac{\varepsilon}{2}$, it follows that $|(-2x+1)-(-1)| < \varepsilon$, which happens to be winning condition for player B.

Since player B's strategy wins no matter how A plays (within the rules :-), it follows from the game definition of limits that $\lim\limits_{x\to 1}(-2x+1) = -1$. $\square$

Solution the second. *(Regular definition)* We'll use the regular ε–δ definition to verify that $\lim\limits_{x\to 1}(-2x+1) = -1$. This means that we have to show that no matter what $\varepsilon > 0$ is given, there is some corresponding $\delta > 0$ such that for any x with $|x-1| < \delta$ we get $|(-2x+1)-(-1)| < \varepsilon$.

Suppose an arbitrary $\varepsilon > 0$ is given. As usual, we attempt to reverse-engineer the $\delta > 0$ from the desired consequence $|(-2x+1)-(-1)| < \varepsilon$:

$$\begin{aligned} |(-2x+1)-(-1)| < \varepsilon \quad &\iff \quad |-2x+1+1| < \varepsilon \quad \iff \quad |-2x+2| < \varepsilon \\ &\iff \quad |-2(x-1)| < \varepsilon \quad \iff \quad |-2|\cdot|x-1| < \varepsilon \\ &\iff \quad 2\,|x-1| < \varepsilon \quad \iff \quad |x-1| < \frac{\varepsilon}{2} \end{aligned}$$

[If this looks familiar, it's because the same engineering process was used in the previous solution ...]

Since every step of this process is reversible, we have that whenever $|x-1| < \delta = \frac{\varepsilon}{2}$, we must also have $|(-2x+1)-(-1)| < \varepsilon$. Thus $\lim\limits_{x\to 1}(-2x+1) = -1$ by the ε–δ difinition of limits. $\square$

2. Use the definition of limit that you didn't use in answering question **1** to verify that $\lim\limits_{x\to 2}(-x+2) \neq 1$. *[2]*

Solution the first. *(Game definition)* To show that $\lim\limits_{x\to 2}(-x+2) \neq 1$ using the game definition, we need to find a winning strategy for player A in the cirresponding limit game. This means finding a particular $\varepsilon > 0$ so that no matter how player B chooses $\delta > 0$, there will be some x with $|x-2| < \delta$ for which $|(-x+2)-1| \geq \varepsilon$ (the winning condition for player A).

The trick here is to choose a convenient $\varepsilon > 0$ that is less than the distance between the supposed limit of 1 and what the function is really doing near $x = 2$: $\lim\limits_{x\to 2}(-x+2) = -2+2 = 0$ because linear functions are continuous. The distance between 1 and 0 is $1-0 = 1$, so we'll have player A choose to play $\varepsilon = \dfrac{1}{2}$ on the first move. Observe that as long as $|x-2| < \frac{1}{2}$, *i.e.* $\frac{3}{2} < x < \frac{5}{2}$, we get $-\frac{1}{2} < -x+2 < \frac{1}{2}$, so $-x+2$ is at least $\varepsilon = \frac{1}{2}$ away from 1.

If player B now plays any $\delta > 0$, player A need only play any x satisfying both $|x-2| < \delta$ (to play within the rules) and $|x-2| < \frac{1}{2}$ (in order to win), that is any x such that $|x-2| < \min\left(\delta, \frac{1}{2}\right)$. Since $|x-2| < \frac{1}{2}$, by the observation above we have $|(-x+2)-1| \geq \frac{1}{2} = \varepsilon$, as required for player A to win.

Since the given strategy wins for player A no matter what player B may do, it follows by the game definition of limits that $\lim_{x\to 2}(-x+2) \neq 1$. □

Solution the second. *(Regular definition)* To show that $\lim_{x\to 2}(-x+2) \neq 1$ using the regular $\varepsilon - \delta$ definition, we need to find a particular $\varepsilon > 0$ such that for every $\delta > 0$ one can find an x with $|x-2| < \delta$ but $|(-x+2)-1| \geq \varepsilon$. As above, the key is to pick an $\varepsilon > 0$ that is smaller than the distance between what $\lim_{x\to 2}(-x+2)$ really is, namely $-2+2=0$, and the alleged limit 1. Since the distance between 1 and 0 is $1-0=1$, $\varepsilon = \frac{1}{2}$ ought to do. Observe that as long as $|x-2| < \frac{1}{2}$, *i.e.* $\frac{3}{2} < x < \frac{5}{2}$, we get $-\frac{1}{2} < -x+2 < \frac{1}{2}$, so $-x+2$ is at least $\varepsilon = \frac{1}{2}$ away from 1.

Now for $\delta > 0$, any x satisfying both $|x-2| < \delta$ and $|x-2| < \frac{1}{2}$, *i.e.* any x with $|x-2| < \min\left(\delta, \frac{1}{2}\right)$, will, by the observation above, have $|(-x+2)-1| \geq \frac{1}{2} = \varepsilon$. It follows by the $\varepsilon - \delta$ definition of limits that $\lim_{x\to 2}(-x+2) \neq 1$. □

Solution the third. *(The indirect approach)* One could also use either version of the definition of limits to show that $\lim_{x\to 2}(-x+2) = 0$ – the details are much like the solutions to question **1** – and then observe that $0 \neq 1$. □

3. Use either definition of limits above to verify that $\lim_{x\to 3}\left(x^2-5\right) = 2$. *[3]*

Hint: The choice of δ in **3** will probably require some slightly indirect reasoning. Pick some arbitrary smallish positive number for δ as a first cut. If it doesn't do the job, but x is at least that close, you'll have more information to help pin down the δ you really need.

Solution. Note first that the limit is incorrect: since polynomials are continuous, $\lim_{x\to 3}\left(x^2-5\right) = 3^2-5 = 9-5 = 4 \neq 2$. The true task is therefore to verify that $\lim_{x\to 3}\left(x^2-5\right) \neq 2$. We will use the usual $\varepsilon - \delta$ definition to show that this is so. We therefore need to find a particular $\varepsilon > 0$ such that for every $\delta > 0$ one can find an x with $|x-3| < \delta$ but $\left|\left(x^2-5\right)-2\right| \geq \varepsilon$. The key is to pick an $\varepsilon > 0$ that is smaller than the distance between what $\lim_{x\to 3}\left(x^2-5\right))$ really is, namely 4, and the alleged limit 1. $4-1=3$ and $\varepsilon = 1$ is a convenient simple number less than this distance.

Fiddling a bit with a calculator lets us observe that as long as $|x-3| < \frac{1}{10} = 0.1$, *i.e.* $2.9 < x < 3.1$, we have $8.41 < x^2 < 9.61$ and so $1.41 < \left(x^2-5\right) - 2 < 2.61$, which guarantees that $\left|\left(x^2-5\right)-2\right| \geq 1.41 > 1 = \varepsilon$. Now, for any $\delta > 0$, if one chooses an x so that both $|x-3| < \delta$ and $|x-3| < 0.1$, *i.e.* so that $|x-3| < \min(\delta, 0.1)$, we get that $\left|\left(x^2-5\right)-2\right| \geq \varepsilon$. Thus, according to the usual $\varepsilon - \delta$ definition of limits, $\lim_{x\to 3}\left(x^2-5\right) \neq 2$.

One could also have solved this problem by finding a winning strategy for player A in the corresponding limit game – the key parts of this would be very similar to parts of the argument above – or indirectly, by verifying that $\lim_{x\to 3}\left(x^2-5\right) = 4$ using either definition of limit and observing that $4 \neq 2$. Taking this approach will require more work than it does for a linear function – see Example 2.3.4 in the textbook for an example of how to handle a quadratic using the usual $\varepsilon - \delta$ definition. $\square$

NOTE: The problems above are probably easiest done by hand, though `Maple` and its competitors do have tools for solving inequalities which could be useful.

4. Use `Maple` (or a program with similar capabilities) to compute $\lim_{x\to 0}\frac{\sin(x+\pi)}{x}$. *[2]*

SOLUTION. After looking up the word "`limit`" in `Maple`'s Help:

```
> limit( sin(x+Pi)/x, x=0 )
```

$$-1$$

Thus `Maple` gives $\lim_{x\to 0}\frac{\sin(x+\pi)}{x} = -1$. $\square$

5. Compute $\lim_{x\to 0}\frac{\sin(x+\pi)}{x}$ by hand. *[1]*

SOLUTION. Note that $\sin(x+\pi) = -\sin(x)$ for all x. One can guess this by comparing the graphs of $y = \sin(x)$ and $y = \sin(x+\pi)$, or prove it by using the general addition formula for $\sin(x)$, namely $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$:

$$\sin(x+\pi) = \sin(x)\cos(\pi) + \cos(x)\sin(\pi) = \sin(x)\cdot(-1) + \cos(x)\cdot 0 = -\sin(x)$$

It follows that $\lim_{x\to 0}\frac{\sin(x+\pi)}{x} = \lim_{x\to 0}\frac{-\sin(x)}{x} = -\lim_{x\to 0}\frac{\sin(x)}{x} = -1$. $\blacksquare$