Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Fall 2019

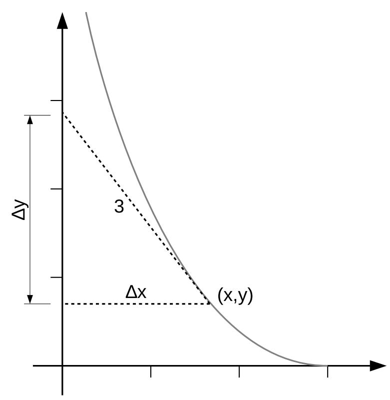
Solutions to Assignment #3
Tracking the Tractor?

There are several instances in mythology of persons being punished for their failings in life by being given a task impossible to complete in their afterlife. For example, in Greek myth Sisyphus is condemned to roll a boulder up a hill, only to have it get away from him and roll back downhill every time he gets it near the top; for another, in Chinese folklore, Wu Gang gets tasked with trying to chop down a tree that regenerates immediately after each strike with his axe.

Our protagonist, Tractor ("Puller"), has been condemned to pull a heavy load. In the beginning, Tractor is at the origin in the Cartesian plane holding one end of a taut rope and the load attached to the other end of the rope is at the point $(3, 0)$. Tractor must walk up the y-axis pulling the load until the load reaches the y-axis. The rope always stays taut, doesn't stretch, shirink, or break, and is always tangent to the curve traced out by the load. For simplicity, we assume that Tractor and the load each occupy no more than a single point each at any given instant. Our main task will be to figure out the equation $y = f(x)$ of the curve traced out by the load.

1. Use the geometry of the situation to find an expression in terms of x for $\frac{dy}{dx} = f'(x)$. Give a sketch of the situation to help explain how you got that expression. *[3]*

SOLUTION. Here's a sketch of the setup:



The curve is the path followed by the load. When the load is at (x, y), the tangent line meets the y-axis at Tractor's location at that instant, exactly 3 units away. This means we have two ways to find the slope of the tangent line.

First, if the curve along which the load moves is given by $y = f(x)$, then the slope of the tangent line at (x, y) is given by $\frac{dy}{dx} = f'(x)$.

Second, the part of the tangent line between (x, y) and the y-axis is 3 units long and is the hypotenuse of the right triangle whose short sides are given by Δx and Δy. It is not hard to see that $\Delta x = x - 0 = x$, and it now follows by the Pythagorean Theorem that $\Delta y = \sqrt{3^2 - (\Delta x)^2} = \sqrt{9 - x^2}$. Thus the slope of the tangent line at (x, y) is

$-\dfrac{\Delta y}{\Delta x} = -\dfrac{\sqrt{9-x^2}}{x}$. Note that we need the negative sign because it is clear from the diagram that the slope is negative, while both x and $\sqrt{9-x^2}$ are positive.

It follows that curve followed by the load satisfies the differential equation:

$$\frac{dy}{dx} = -\frac{\sqrt{9-x^2}}{x}$$

The other piece of information in the setup that we'll need to solve this equation is the fact that the curve passes through the point $(3,0)$, *i.e.* $y = f(0) = 3$ when $x = 0$. □

2. Use `Maple` (or an equivalent) to use the given information and your answer to question **1** to find $y = f(x)$. (Please provide a printout of your worksheet.) *[4]*

Hint: You'll probably want to look up the `diff` operator and the `dsolve` command if using `Maple`.

Solution. We type into a `Maple` worksheet and hope for the best:

```
> dsolve({diff(y(x),x)=-sqrt(9-x^2)/x,y(3)=0},y(x));
```

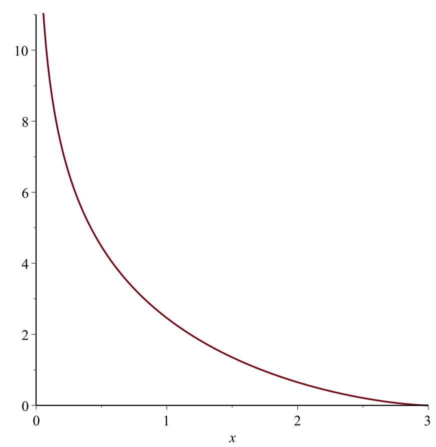
$$y(x) = -\sqrt{-x^2+9} + 3\operatorname{arctanh}\left(\frac{3}{\sqrt{-x^2+9}}\right) + \frac{3}{2}\mathrm{I}\pi$$

This solution, if you think about it, is a bit problematic: the "I" in the constant term represents the "imaginary" number $i = \sqrt{-1}$. The solution is nevertheless correct, though showing that it is equal to the one that would usually be obtained by hand (with techniques you would see in MATH 1120H) takes a lot of work. □

3. Use `Maple` (or an equivalent) to plot $y = f(x)$. *[1]*

Solution. Here goes:

```
> plot(-sqrt(-x^2+9)+3*arctanh(3/sqrt(-x^2+9))+(3/2)*I*Pi,x=0..3)
```



□

4. Does Tractor ever get to stop, or must the load be pulled forever? *[2]*

Solution. As is apparent from the plot, the curve followed by the load has a vertical asymptote at $x = 0$, so it will never actually reach the y-axis. Sadly, this means that Tractor must pull the load for all eternity. ■