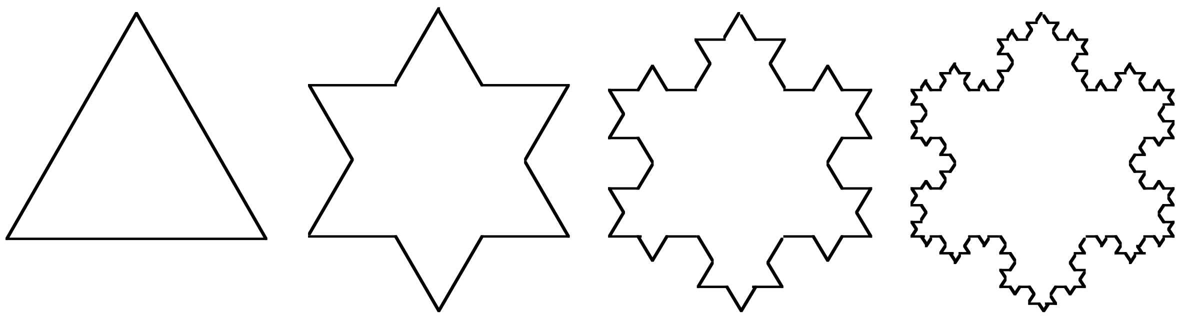
Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
Trent University, Fall 2018

Solutions to Assignment #6
The ultimate snowflake - because winter is coming!

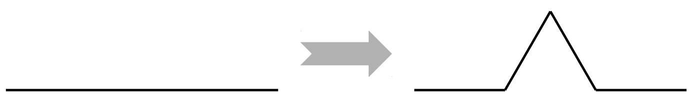
Suppose one starts with an equilateral triangle with sides of length 1. If one modifies each of the line segments composing the triangle by cutting out the middle third of the segment, and then inserting an outward-pointing "tooth," both of whose sides are as long as the removed third, one gets a six-pointed star. Repeating this for each of the line segments making up the star, then to each of the line segments making up the resulting figure, and so on, leads gives increasingly intricate curves with more and more corners. The first stages of the process are pictured in the diagram below:



Note that the lengths of the line segments at each stage are a third of the length of the segments at the preceding stage. For the sake of being definite, let's say we have the triangle at step 0 of the process, the six-pointed star at step 1 of the process, the next shape at step 2 of the process, and so on. The curve which is the limit of this process, if one takes infinitely many steps, is often called the *snowflake curve**. We will investigate the length of this curve and the area of the region that it encloses.

1. Find the length of the snowflake curve. *[5]*

Solution. At each stage of the process all you're doing is replacing each line segment that is part of the perimeter by a "toothed" segment made up of four pieces, each one third as long as the original:



It should be blindingly obvious that this will increase the length of the perimeter by $\frac{4}{3}$ at each stage of the process. Thus, if you started at stage 0 with an equilateral triangle which has perimeter 1, the next stage would have a perimeter of $\frac{4}{3}$, the second stage would

* Also known as the *Koch curve*. Note that "curve" does not imply smoothness here: although it is continuous, the snowflake curve has infinitely many corners and is therefore none too differentiable ...

have a perimeter of $\frac{4}{3} \cdot \frac{4}{3} = \left(\frac{4}{3}\right)^2$, and so on. In general, the nth stage in the process has a perimeter of $\left(\frac{4}{3}\right)^n$.

Since the snowflake curve is the limit of the shapes at each stage, the perimeter of the snowflake curve is the limit of the perimeters at each stage, namely $\displaystyle\lim_{n\to\infty} \left(\frac{4}{3}\right)^n = \infty$. (Why ∞? *You* figure it out!) Thus the snowflake curve has an infinite perimeter. ■

2. Find the area of the region enclosed by the snowflake curve. *[5]*

Solution. More than anything else, this is an exercise in counting.

First, observe that at stage 0, our shape – the equilateral triangle – has a border made up of three line segments, each of length 1. Going from one stage to the next, each line segment is replaced by four line segments, each of length $\frac{1}{3}$ of those at the previous stage. Thus at stage 1, the perimeter of the six-pointed star is made up of $3 \cdot 4 = 12$ line segments which are each of length $\frac{1}{3} \cdot 1 = \frac{1}{3}$, at stage 2, the perimeter of the shape is made up of $(3 \cdot 4) \cdot 4 = 3 \cdot 4^2 = 48$ line segments which are each of length $\frac{1}{3} \cdot \frac{1}{3} = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$, and so on. If we denote the number of line segments at stage n by a_n and the the length of eah line segment at stage n by b_n, it follows that $a_n = 3 \cdot 4^n$ and $b_n = \left(\frac{1}{3}\right)^n = \frac{1}{3^n}$.

Second, the area of an equilateral triangle with sides of length b is $\frac{\sqrt{3}}{4}b^2$. To see this, drop an altitude from one of the vertices to the opposite side. This divides the triangle into two congruent right triangles, each with a hypotenuse of length b and short sides of length $\frac{b}{2}$ and the altitude. It follows from the Pythagorean Theorem that the altitude has length $\sqrt{b^2 - \left(\frac{b}{2}\right)^2} = \sqrt{b^2 - \frac{b^2}{4}} = \sqrt{\frac{3}{4}b^2} = \frac{\sqrt{3}}{2}b$. Thus the area of the equilateral triangle is $\frac{1}{2}b \cdot \frac{\sqrt{3}}{2}b = \frac{\sqrt{3}}{4}b^2$. For example, the equilateral triangle with sides of length 1 we have at stage 0 has an area of $\frac{\sqrt{3}}{4}1^2 = \frac{\sqrt{3}}{4}$.

Third, at stage $n > 0$ of the process for making the snowflake curve we enclose all the area that was enclosed at the preceding stage and add the area of all the small triangles added at stage n. How many such triangles are there? Just as many as there were line segments at the previous stage, that is $a_{n-1} = 3 \cdot 4^{n-1}$. What is the area of each? Combining information from the previous two paragraphs, the area of each new triangle is $\frac{\sqrt{3}}{4}b_n^2 = \frac{\sqrt{3}}{4}\left(\frac{1}{3}\right)^{2n}$. It follows that at stage $n > 0$ we add an area of $3 \cdot 4^{n-1} \cdot \dfrac{\sqrt{3}}{4}\left(\dfrac{1}{3}\right)^{2n} = \dfrac{\sqrt{3}}{4} \cdot \dfrac{4^{n-1}}{3^{2n-1}}$ to the area we already had at the previous stage.

Fourth, it follows from the observations above that the total area of the shape at stage $n > 0$ of the process is:

$$\begin{aligned}
&\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \cdot \frac{4^{1-1}}{3^{2\cdot 1-1}} + \frac{\sqrt{3}}{4} \cdot \frac{4^{2-1}}{3^{2\cdot 2-1}} + \cdots + \frac{\sqrt{3}}{4} \cdot \frac{4^{n-1}}{3^{2n-1}} \\
&= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\left[\frac{4^0}{3^1} + \frac{4^1}{3^3} + \cdots + \frac{4^{n-1}}{3^{2n-1}}\right] = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4 \cdot 3}\left[\frac{4^0}{3^0} + \frac{4^1}{3^2} + \cdots + \frac{4^{n-1}}{3^{2n-2}}\right] \\
&= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}\left[1 + \frac{4}{9} + \cdots + \left(\frac{4}{9}\right)^{n-1}\right] = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \cdot \frac{1 - \left(\frac{4}{9}\right)^n}{1 - \frac{4}{9}}
\end{aligned}$$

In the last step, we used the formula for the sum of a finite geometric series, $a + ar + ar^2 + \cdots + ar^k = a\frac{1-r^{k+1}}{1-r}$, which we encountered in Assignment #0. (As noted in the solutions to Assignment #0, geometric series and their summation formulas can be found in Example 11.2.1 in the textbook.)

Finally, since the shape enclosed by the snowflake curve is the limit of the shapes at the various stages, and because $\left(\frac{4}{9}\right)^n \to 0$ as $n \to \infty$, the area enclosed by the snowflake curve is:

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}\cdot\frac{1-\left(\frac{4}{9}\right)^n}{1-\frac{4}{9}}\right) &= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}\cdot\frac{1-0}{1-\frac{4}{9}} = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}\cdot\frac{1}{\frac{5}{9}} \\ &= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}\cdot\frac{9}{5} = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\cdot\frac{1}{3}\cdot\frac{9}{5} \\ &= \frac{\sqrt{3}}{4}\left[1+\frac{3}{5}\right] = \frac{\sqrt{3}}{4}\cdot\frac{8}{5} = \frac{2\sqrt{3}}{5}\end{aligned}$$

Whew! ■