Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Fall 2018

Solutions to Assignment #0[‡]
Limitations and Summations

Suppose we let $a_n = \frac{1}{3} + \frac{1}{4} + \frac{3}{16} + \frac{9}{64} + \cdots + \frac{3^{n-1}}{4^n}$ for $n = 0, 1, 2, \dots$ That is, we have $a_0 = \frac{1}{3}$, $a_1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$, $a_2 = \frac{1}{3} + \frac{1}{4} + \frac{3}{16} = \frac{37}{48}$, and so on.

Hint: Look up geometric series.

DISCUSSION: Looking up "geometric series" in our textbook's index we see that this term is mentioned on page 262. Near the bottom of page 262, we find Example 11.2.1 which tells us that a geometric series is a series – that is, an infinite sum, as defined previously on page 262 – of the form

$$k + kx + kx^2 + \cdots = \sum_{n=0}^{\infty} kx^n \,,$$

where k and x are numbers. (k is the *initial value* and x is the *common ratio* between successive terms.) Reading through Example 11.2.1 we discover that the nth *partial sum* of the series (*i.e.* a finite geometric series) has a nice formula,

$$s_n = k + kx + kx^2 + \cdots + kx^n = \frac{k\left(1 - x^{n+1}\right)}{1-x} \,,$$

Note that while the notation a_n as used in Example 11.2.1 represents one of the individual numbers being added up, in the statement of the set-up for this assignment, a_n is a partial sum of the numbers being added up. As

$$a_n = \frac{1}{3} + \frac{1}{4} + \frac{3}{16} + \frac{9}{64} + \cdots + \frac{3^{n-1}}{4^n} = \frac{1}{3} + \frac{1}{3} \cdot \frac{3}{4} + \frac{1}{3} \cdot \left(\frac{3}{4}\right)^2 + \cdots + \frac{1}{3} \cdot \left(\frac{3}{4}\right)^n \,,$$

a_n is actually the nth partial sum of the geometric series with initial value $k = \frac{1}{3}$ and common ratio $x = \frac{3}{4}$. By the formula noted above it follows that:

$$a_n = \frac{\frac{1}{3}\left(1 - \left(\frac{3}{4}\right)^{n+1}\right)}{1 - \frac{3}{4}} = \frac{1}{3} \cdot \frac{1 - \left(\frac{3}{4}\right)^{n+1}}{\frac{1}{4}} = \frac{4}{3} \cdot \left(1 - \left(\frac{3}{4}\right)^{n+1}\right)$$

Example 11.2.1 also tells us that the entire series also has a nice formula – well, so long as $|x| < 1$ – namely,

$$\sum_{n=0}^{\infty} kx^n = k + kx + kx^2 + \cdots = \frac{k}{1-x} \,.$$

[‡] Think of it as bonus, or perhaps as a warmup, assignment. There is less to it than meets the eye, especially if you take the hint.

(The latter formula fails spectacularly if $|x| \geq 1$, by the way; for example, with $k = 1$ and $x = 2$ you get that $1 + 2 + 4 + 8 + \cdots = \frac{1}{1-2} = -1$, which is just a bit absurd.) In our case, since we have $k = \frac{1}{3}$ and $x = \frac{3}{4}$ – note that $\left|\frac{3}{4}\right| = \frac{3}{4} < 1$ – we have:

$$\frac{1}{3} + \frac{1}{4} + \frac{3}{16} + \frac{9}{64} + \cdots = \frac{1}{3} + \frac{1}{3} \cdot \frac{3}{4} + \frac{1}{3} \cdot \left(\frac{3}{4}\right)^2 + \cdots = \frac{\frac{1}{3}}{1 - \frac{3}{4}} = \frac{\frac{1}{3}}{\frac{1}{4}} = \frac{4}{3}$$

After wading through all of that, the given problems are pretty easy:

1. Compute $L = \lim_{n\to\infty} a_n$. *[5]*

SOLUTION. Using the formula for a_n noted above:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4}{3} \cdot \left(1 - \left(\frac{3}{4}\right)^{n+1}\right) = \frac{4}{3} \cdot (1 - 0) = \frac{4}{3},$$

since $\left(\frac{3}{4}\right)^{n+1} \to 0$ as $n \to \infty$ because $\left|\frac{3}{4}\right| < 1$. Alternatively, you could observe that:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{1}{3} + \frac{1}{3} \cdot \frac{3}{4} + \cdots + \frac{1}{3} \cdot \left(\frac{3}{4}\right)^n\right) = \frac{1}{3} + \frac{1}{3} \cdot \frac{3}{4} + \frac{1}{3} \cdot \left(\frac{3}{4}\right)^2 + \cdots = \frac{4}{3}$$

Either way, the limit exists and is equal to $L = \frac{4}{3}$. ■

2. How large does N have to be to guarantee that a_n is within $0.01 = \dfrac{1}{100}$ of L for *all*l $n \geq N$? Explain why! *[5]*

SOLUTION. Note that $|L - a_n| = L - a_n = \dfrac{4}{3} - \dfrac{4}{3} \cdot \left(1 - \left(\dfrac{3}{4}\right)^{n+1}\right) = \dfrac{4}{3} \cdot \left(\dfrac{3}{4}\right)^{n+1} = \left(\dfrac{3}{4}\right)^n$.

We want to have $|L - a_n| = \left(\dfrac{3}{4}\right)^n < 0.01$. A little work with a calculator reveals that $\left(\frac{3}{4}\right)^{16} \approx 0.01002$ and $\left(\frac{3}{4}\right)^{17} \approx 0.00752$.

$N = 17$ will do: if $n \geq N = 17$, then $\left(\dfrac{3}{4}\right)^n = \left(\dfrac{3}{4}\right)^{17} \cdot \left(\dfrac{3}{4}\right)^{n-17} \leq \left(\dfrac{3}{4}\right)^{17} < 0.01$ because $\left(\dfrac{3}{4}\right)^{n-17} \leq 1$, as $\frac{3}{4} < 1$ and $n - 17 \geq 0$. ■

NOTE. Those who don't like the experimentation with a cal;culator in **2**, can always get the job done with the help of logarithms. Of course, you're likely to need a calculator to compute those ...