Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals

TRENT UNIVERSITY, Fall 2018

The Solutions to the Final ~~Countdown~~ Examination

Space-time: *Gym – 11:00-14:00.* *Brought to you by* Стефан Біланюк.

Instructions: Do parts □ and △, and, if you wish, part ○. Please show all your work and justify all your answers. *If in doubt about something,* **ask!**

Aids: Any calculator; (all sides of) one aid sheet; one (1) brain (no neuron limit).

Part □. Do all four (4) of **7**–**4**. *[Subtotal = 74]*

7. Compute $\frac{dy}{dx}$ as best you can in any *four* (4) of **f**–**a**. *[20 = 4 × 5 each]*

$$\textbf{f. } y=(x-1)e^x \quad \textbf{e. } \ln(y-x)=0 \quad \textbf{d. } y=\int_0^{\sqrt{x}} \cos(t)\,dt$$

$$\textbf{c. } y=\frac{x-1}{x+1} \quad \textbf{b. } y=\tan\left(e^{2x}\right) \quad \textbf{a. } y=x^\pi+x^e+2018$$

SOLUTIONS. **f.** This is job for the Product Rule:

$$\frac{dy}{dx}=\frac{d}{dx}(x-1)e^x=\left[\frac{d}{dx}(x-1)\right]e^x+(x-1)\left[\frac{d}{dx}e^x\right]=1e^x+(x-1)e^x=xe^x \quad \square$$

e. $\ln(y-x)=0 \Rightarrow y-x=1 \Rightarrow 7=x+1 \Rightarrow \frac{dy}{dx}=\frac{d}{dx}(x+1)=1 \quad \square$

d. The Fundamental Theorem of Calculus and the Chain Rule:

$$\frac{dy}{dx}=\frac{d}{dx}\int_0^{\sqrt{x}}\cos(t)\,dt=\cos\left(\sqrt{x}\right)\cdot\frac{d}{dx}\sqrt{x}=\cos\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}} \quad \square$$

c. Quotient Rule:

$$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}\left(\frac{x-1}{x+1}\right)=\frac{\left[\frac{d}{dx}(x-1)\right](x+1)-(x-1)\left(\frac{d}{dx}(x+1)\right]}{(x+1)^2}\\&=\frac{1(x+1)-(x-1)1}{(x+1)^2}=\frac{2}{(x+1)^2} \quad \square\end{aligned}$$

b. Chain Rule all the way:

$$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}\tan\left(e^{2x}\right)=\sec^2\left(e^{2x}\right)\cdot\frac{d}{dx}e^{2x}=\sec^2\left(e^{2x}\right)\cdot e^{2x}\cdot\frac{d}{dx}(2x)\\&=2e^{2x}\sec^2\left(e^{2x}\right) \quad \square\end{aligned}$$

a. Power Rule: $\frac{dy}{dx}=\frac{d}{dx}\left(x^\pi+x^e+2018\right)=\pi x^{\pi-1}+ex^{e-1}$ ■

6. Evaluate any *four* (4) of the integrals **f**–**a**. *[20 = 4 × 5 each]*

$$\textbf{f.}\ \int e^x\cos(x)\,dx \quad \textbf{e.}\ \int_1^2 \frac{w^2-w-2}{w+1}\,dw \quad \textbf{d.}\ \int_0^{\pi/4} 2\tan(z)\sec^2(z)\,dz$$

$$\textbf{c.}\ \int \frac{e^{\sqrt{y}}}{\sqrt{y}}\,dy \qquad \textbf{b.}\ \int t\cosh(t)\,dt \qquad \textbf{a.}\ \int_0^{\pi/2} \frac{\cos(u)}{1+\sin^2(u)}\,du$$

SOLUTIONS. **f.** We will use integration by parts twice, followed by a bit of algebra. The first use of parts will have $u = e^x$ and $v' = \cos(x)$, so $u' = e^x$ and $v = \sin(x)$; the second will have $s = e^x$ and $t' = \sin(x)$, so $s' = e^x$ and $t = -\cos(x)$.

$$\begin{aligned}\int e^x\cos(x)\,dx &= e^x\sin(x) - \int e^x\sin(x)\,dx \\ &= e^x\sin(x) - \left[e^x\left(-\cos(x)\right) - \int e^x\left(-\cos(x)\right)\,dx\right] \\ &= e^x\sin(x) + e^x\cos(x) - \int e^x\cos(x)\,dx\end{aligned}$$

We now can solve for our integral. By bringing the copy of the integral on the right-hand side to the left-hand side, we have $2\int e^x\cos(x)\,dx = e^x\sin(x) + e^x\cos(x)$. It follows that $\int e^x\cos(x)\,dx = \frac{1}{2}\left[e^x\sin(x) + e^x\cos(x)\right] + C$. ☐

e. A little algebra, followed by the Power Rule:

$$\begin{aligned}\int_1^2 \frac{w^2-w-2}{w+1}\,dw &= \int_1^2 \frac{(w-2)(w+1)}{w+1}\,dw = \int_1^2 (w-2)\,dw = \left.\left(\frac{w^2}{2} - 2w\right)\right|_1^2 \\ &= \left[\frac{2^2}{2} - 2\cdot 2\right] - \left[\frac{1^2}{2} - 2\cdot 1\right] = [2-4] - \left[\frac{1}{2} - 2\right] \\ &= [-2] - \left[-\frac{3}{2}\right] = -\frac{1}{2} \quad \square\end{aligned}$$

d. We will use the substitution $u = \tan(z)$, so $\frac{du}{dz} = \sec^2(z)$ and thus $du = \sec^2(z)\,dz$, and change the limits as we go along: $\begin{matrix} x & 0 & \pi/4 \\ u & 0 & 1 \end{matrix}$.

$$\int_0^{\pi/4} 2\tan(z)\sec^2(z)\,dz = \int_0^1 2u\,du = \left.u^2\right|_0^1 = 1^2 - 0^2 = 1 \quad \square$$

c. We will use the substitution $w = \sqrt{y}$, so $\frac{dw}{dy} = \frac{1}{2\sqrt{y}}$, giving $dw = \frac{1}{2\sqrt{y}}\,dy$ and thus $2\,dw = \frac{1}{\sqrt{y}}\,dy$.

$$\int \frac{e^{\sqrt{y}}}{\sqrt{y}}\,dy = \int e^w\cdot 2\,dw = 2e^w + C = 2e^{\sqrt{y}} + C \quad \square$$

b. We will use integration by parts, with $u = t$ and $v' = \cosh(t)$, so $u' = 1$ and $v = \sinh(t)$.

$$\int t\cosh(t)\,dt = t\sinh(t) - \int 1\sinh(t)\,dt = t\sinh(t) - \cosh(t) + C \quad \Box$$

a. We will use the substitution $w = \sin(u)$, so $\dfrac{dw}{du} = \cos(u)$ and hence $dw = \cos(u)\,du$, and change the limits as we go along, $\begin{matrix} u & 0 & \pi/2 \\ w & 0 & 1 \end{matrix}$.

$$\begin{aligned}\int_0^{\pi/2} \frac{\cos(u)}{1+\sin^2(u)}\,du &= \int_0^1 \frac{1}{1+w^2}\,dw = \arctan(w)\big|_0^1 \\ &= \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4} \quad \blacksquare\end{aligned}$$

5. Do any *four* (4) of **a**–**f**. *[20 = 4 × 5 each]*

f. Find the equation of the tangent line to $y = \tan(x)$ at $x = 0$.

e. Use the ε–δ definition of limits to verify that $\lim\limits_{x \to -1} (2x+3) = 1$.

d. Use the limit definition of the derivative to verify that $\dfrac{d}{dx}x^3 = 3x^2$ for all x.

c. Find the minimum value of $f(x) = x^3 - x^2 + x$ on the interval $[0, 2]$.

b. Compute $\lim\limits_{x \to \infty} \dfrac{e^x}{e^{2x}+1}$.

a. Find the area of the the region between $y = \cos(x)$ and $y = \sin(x)$ for $0 \le x \le \dfrac{\pi}{2}$.

SOLUTIONS. **f.** The slope of the tangent line is:

$$m = \left.\frac{dy}{dx}\right|_{x=0} = \left.\frac{d}{dx}\tan(x)\right|_{x=0} = \sec^2(x)\big|_{x=0} = \sec^2(0) = 1^2 = 1$$

Since $0 = \tan(0) = m \cdot 0 + b = 1 \cdot 0 + b = b$, it follows that the equation of the tangent line to $y = \tan(x)$ at $x = 0$ is $y = 1x + 0 = x$. $\Box$

e. We need to check that for every $\varepsilon > 0$ there is a $\delta > 0$ such that every x with $|x - (-1)| < \delta$ yields $|(2x+3) - 1| < \varepsilon$. As usual, we reverse-engineer the δ from the ε:

$$|(2x+3) - 1| < \varepsilon \Leftrightarrow |2x+2| < \varepsilon \Leftrightarrow 2|x+1| < \varepsilon \Leftrightarrow |x - (-1)| < \frac{\varepsilon}{2}$$

Since every step above was reversible, if we are given a $\varepsilon > 0$, setting $\delta = \frac{\varepsilon}{2}$ will ensure that whenever $|x - (-1)| < \delta$, we get $|(2x+3) - 1| < \varepsilon$, as required.

It follows that $\lim\limits_{x \to -1} (2x+3) = 1$ by the ε–δ definition of limits. $\Box$

d. The limit definition of the derivative of $f(x)$ is $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$. We apply this to $f(x) = x^3$ and evaluate the limit:

$$\begin{aligned}\frac{d}{dx}x^3 &= \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}\\ &= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h} = \lim_{h\to 0}\left(3x^2+3xh+h^2\right)\\ &= 3x^2+3x\cdot 0+0^2 = 3x^2 \quad \square\end{aligned}$$

c. We need to find any critical points of $f(x) = x^3 - x^2 + x$ in the interval $[0,2]$ and compare the value(s) of $f(x)$ at any such points with its values at the endpoints of the interval.

First, $f'(x) = \dfrac{d}{dx}\left(x^3 - x^2 + x\right) = 3x^2 - 2x + 1$. According to the quadratic formula, this equals 0 when $x = \dfrac{-(-2)\pm\sqrt{(-2)^2-4\cdot 3\cdot 1}}{2} = \dfrac{2\pm\sqrt{4-12}}{2} = \dfrac{2\pm\sqrt{-8}}{2}$. Since $\sqrt{-8}$ is not a real number, $f(x)$ has no critical points for us to check.

Second, we still have to check the values of $f(x)$ on the endpoints of the interval $[0,2]$: $f(0) = 0^3 - 0^2 + 0 = 0$ and $f(2) = 2^3 - 2^2 + 2 = 8 - 4 + 2 = 6$. Since $0 < 6$, it follows that the minimum value of $f(x) = x^3 - x^2 + x$ on the interval $[0,2]$ is $f(0) = 0$. $\square$

b. One could do this with l'Hôpital's Rule and some algebra, but it's faster with just some algebra: $\displaystyle\lim_{x\to\infty}\frac{e^x}{e^{2x}+1} = \lim_{x\to\infty}\frac{e^x}{e^{2x}+1}\cdot\frac{e^{-x}}{e^{-x}} = \lim_{x\to\infty}\frac{1}{e^x+e^{-x}}\,\frac{\to 1}{\to\infty+0} = 0.$ $\square$

a. Note that $y = \cos(x)$ and $y = \sin(x)$ intersect once for some $0 \le x \le \dfrac{\pi}{2}$, namely at $x = \dfrac{\pi}{4}$. Since $\sin(0) = 0 < 1 = \cos(0)$ and $\sin\left(\frac{\pi}{2}\right) = 1 > 0 - \cos\left(\frac{\pi}{2}\right)$, it follows that $y = \cos(x)$ is above $y = \sin(x)$ for $0 \le x < \dfrac{\pi}{4}$ and below it for $\dfrac{\pi}{4} < x \le \dfrac{\pi}{2}$. The area of the region between the curves is therefore given by:

$$\begin{aligned}\int_0^{\pi/2}(\text{upper}-\text{lower})\,dx &= \int_0^{\pi/4}(\cos(x)-\sin(x))\,dx + \int_{\pi/4}^{\pi/2}(\sin(x)-\cos(x))\,dx\\ &= [\sin(x)-(-\cos(x))]\Big|_0^{\pi/4} + [(-\cos(x))-\sin(x)]\Big|_{\pi/4}^{\pi/2}\\ &= [\sin(x)+\cos(x)]\Big|_0^{\pi/4} - [\sin(x)+\cos(x)]\Big|_{\pi/4}^{\pi/2}\\ &= \left[\sin\left(\frac{\pi}{4}\right)+\cos\left(\frac{\pi}{4}\right)\right] - [\sin(0)+\cos(0)]\\ &\qquad - \left(\left[\sin\left(\frac{\pi}{2}\right)+\cos\left(\frac{\pi}{2}\right)\right] - \left[\sin\left(\frac{\pi}{4}\right)+\cos\left(\frac{\pi}{4}\right)\right]\right)\\ &= \left[\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right] - [0+1] - \left([1+0] - \left[\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right]\right)\\ &= \frac{2}{\sqrt{2}} - 1 - 1 + \frac{2}{\sqrt{2}} = \frac{4}{\sqrt{2}} - 2 = 2\sqrt{2} - 2 = 2\left(\sqrt{2}-1\right) \quad \blacksquare\end{aligned}$$

4. Find the domain and any and all intercepts, vertical and horizontal asymptotes, intervals of increase, decrease and concavity, and maximum, minimum, and inflection points of $f(x) = \dfrac{x^2}{x^2+1}$, and sketch its graph. *[14]*

Solution. We run through the checklist:

i. (Domain) Both x^2 and $1+x^2$ are defined for all x and, as $1+x^2 \geq 1 > 0$ for all x, $f(x) = \dfrac{x^2}{x^2+1}$ is also defined for all x. Thus the domain of $f(x)$ is $\mathbb{R} = (-\infty, \infty)$.

ii. (Intercepts) $f(0) = \dfrac{0^2}{0^2+1} = 0$ so $f(x)$ has y-intercept 0. Moreover, $= \dfrac{x^2}{x^2+1} = 0$ exactly when $x^2 = 0$, which is to say exactly when $x = 0$, so $x = 0$ also gives the only x-intercept.

iii. (Asymptotes) Since $= \dfrac{x^2}{x^2+1}$ is a rational function, it is continuous wherever it is defined and, as noted above, it is defined for all $x \in \mathbb{R}$. As it is continuous for all x, it follows that it has no vertical asymptotes. It remains to check for horizontal asymptotes:

$$\lim_{x\to-\infty} \frac{x^2}{x^2+1} = \lim_{x\to-\infty} \frac{x^2}{x^2+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \frac{1}{1+\frac{1}{x^2}} \begin{matrix} \to 1 \\ \to 1+0 \end{matrix} = \frac{1}{1+0} = 1$$

$$\lim_{x\to+\infty} \frac{x^2}{x^2+1} = \lim_{x\to+\infty} \frac{x^2}{x^2+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to+\infty} \frac{1}{1+\frac{1}{x^2}} \begin{matrix} \to 1 \\ \to 1+0 \end{matrix} = \frac{1}{1+0} = 1$$

Thus $f(x)$ has a horizontal asymptote of $y = 1$ in both directions.

iv. (Increase/decrease/max/min) We'll need the derivative:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{x^2}{x^2+1}\right) = \frac{\left[\frac{d}{dx}x^2\right]\left(x^2+1\right) - x^2\left[\frac{d}{dx}\left(x^2+1\right)\right]}{\left(x^2+1\right)^2} \\ &= \frac{2x\left(x^2+1\right) - x^2 \cdot 2x}{\left(x^2+1\right)^2} = \frac{2x^3+2x-2x^3}{\left(x^2+1\right)^2} = \frac{2x}{\left(x^2+1\right)^2} \end{aligned}$$

Since $\left(x^2+1\right)^2$ is positive for all x, the derivative will be positive, negative, or zero exactly when the numerator $2x$ is positive, negative, or zero. Thus $f(x)$ is increasing when $f'(x) > 0$, *i.e.* when $x \in (0, \infty)$, and decreasing when $f'(x) < 0$, *i.e.* when $x \in (-\infty, 0)$; it follows that the critical point at $x = 0$ is a minimum. (Note that $f(0) = 0$, as noted in part *ii* above.) We summarize all this in a table:

x	$(-\infty, 0)$	0	$(0, \infty)$
$f'(x)$	$-$	0	$+$
$f(x)$	$\downarrow$	min	$\uparrow$

v. (Curvature/inflection) This time we'll need the second derivative:

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}\left(\frac{2x}{\left(x^2+1\right)^2}\right) = \frac{\left[\frac{d}{dx}2x\right]\left(x^2+1\right)^2 - 2x\left[\frac{d}{dx}\left(x^2+1\right)^2\right]}{\left[\left(x^2+1\right)^2\right]^2}$$

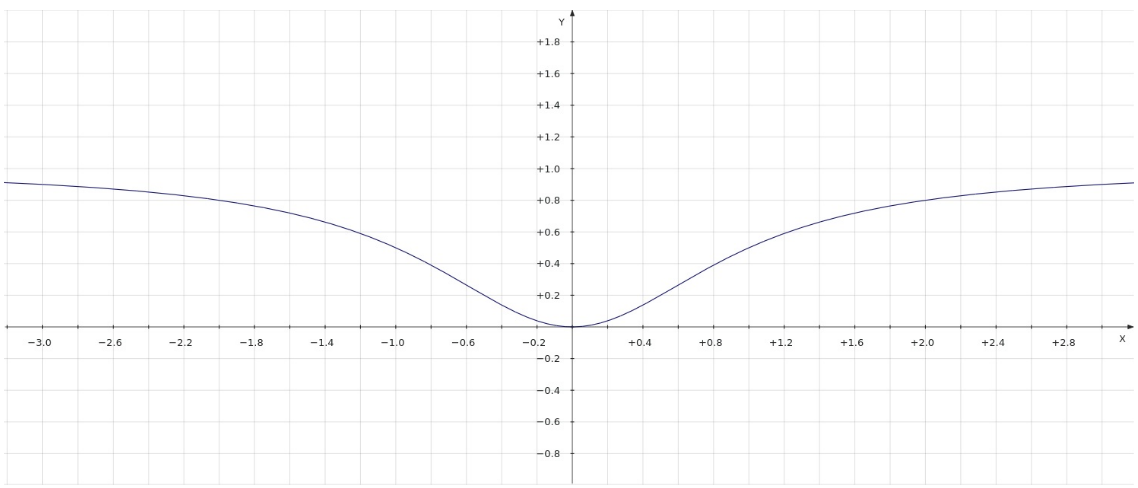
$$= \frac{2\left(x^2+1\right)^2 - 2x\cdot 2\left(x^2+1\right)\left[\frac{d}{dx}\left(x^2+1\right)\right]}{\left(x^2+1\right)^4} = \frac{2\left(x^2+1\right) - 4x \cdot 2x}{\left(x^2+1\right)^3}$$

$$= \frac{2x^2+2-8x^2}{\left(x^2+1\right)^3} = \frac{2-6x^2}{\left(x^2+1\right)^3}$$

Since $\left(x^2+1\right)^3$ is positive for all x (because x^2+1 is), $f''(x)$ is positive, negative, or zero exactly as the numerator $2-6x^2$ is positive, negative, or zero. $2-6x^2 = 2\left(1-3x^2\right)$ is positive exactly when $1-3x^2 > 0$, *i.e.* when $-\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}}$, and negative when $1-3x^2 < 0$. *i.e.* when $x < -\frac{1}{\sqrt{3}}$ or $x > \frac{1}{\sqrt{3}}$; it follows that $f(x)$ has inflection points at $x = -\dfrac{1}{\sqrt{3}}$ and $x = \frac{1}{\sqrt{3}}$. We summarize this in another table:

x	$\left(-\infty, \frac{1}{\sqrt{3}}\right)$	$-\frac{1}{\sqrt{3}}$	$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$\frac{1}{\sqrt{3}}$	$\left(\frac{1}{\sqrt{3}}, \infty\right)$
$f''(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\frown$	inflection	$\smile$	inflection	$\frown$

vi. (Graph) Cheating a bit by having a computer do the job ...



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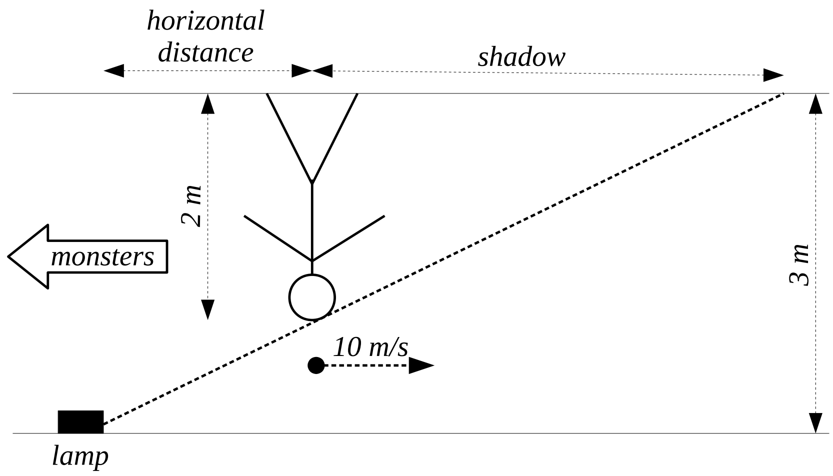
More on the next page!

Part Δ. Do any *two* (2) of **3**–**1**. *[Subtotal = 26]*



3. Jed Aye, who is 2 m tall, explores a 3 m tall tunnel that runs horizontally, walking on the ceiling to avoid traps and carrying a lamp. On spotting monsters farther down the tunnel, Jed drops the lamp in shock and awe and begins running away from the monsters, still on the ceiling, at 10 m/s. At the instant that Jed is a horizontal distance of 10 m away from where the still-functioning lamp landed on the floor, how is the length of Jed's shadow on the ceiling changing with time? *[13]*

SOLUTION. Suppose we denote the horizontal distance between Jed and the lamp by h and the length of Jed's shadow on the ceiling by s. We wish to determine $\frac{ds}{dt}$ at the instant that $h = 10\ m$, given that $\frac{dh}{dt} = 10\ m/s$. To do so we need to sort out the relationship between h and s. Note that the 2 m tall Jed and Jed's shadow are the short sides of a right triangle. This triangle is similar to the larger right triangle whose short side corresponding to Jed is the 3 m perpendicular distance from the fallen lamp to the ceiling and whose short side corresponding to Jed's shadow consists of the horizontal distance along the ceiling between the lamp and Jed, together with Jed's shadow. Since the triangles are similar (since each has a right angle and they share a common angle at the tip of Jed's shadow), corresponding sides have the same ratios. This tells us that $\frac{h+s}{3} = \frac{s}{2}$. We'll use this relation to solve for s in terms of h:

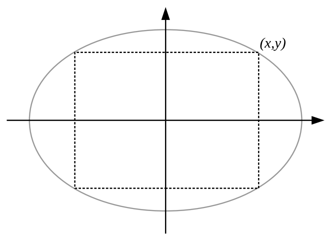
$$\frac{h+s}{3} = \frac{s}{2} \Rightarrow \frac{h}{3} = \frac{s}{2} - \frac{s}{3} = \frac{s}{6} \Rightarrow s = 6 \cdot \frac{h}{3} = 2h$$

It follows that, irrespective of whether $h = 10\ m$ or not,

$$\frac{ds}{dt} = \frac{d}{dt}(2h) = 2\frac{dh}{dt} = 2 \cdot 10 = 20\ m/s\,. \quad \blacksquare$$

2. What is the maximum area of a rectangle which has each side parallel to one of the axes and all of its corners on the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$? *[13]*

Solution. Suppose the upper right corner of the rectangle is at (x, y), as in the diagram below.



Since the ellipse is symmetric about the origin, we have that $0 \leq x \leq 3$ and the other corners of the rectangle, going around counterclockwise from (x, y), are at $(-x, y)$, $(-x, -y)$, and $(x, -y)$. It is not hard to see that this rectangle has width $2x$ and height $2y$, so it has area $A = 4xy$. Since the ellipse has equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and (x, y) is on the ellipse, we also have $y = 2\sqrt{1 - \frac{x^2}{9}}$. It follows that we need to maximize $A(x) = 8x\sqrt{1 - \frac{x^2}{9}}$ for $0 \leq x \leq 3$.

First, note that $A(0) = A(3) = 0$. We stlll need to find the critical points of $A(x)$ in $[0, 3]$ and check what $A(x)$ is at such.

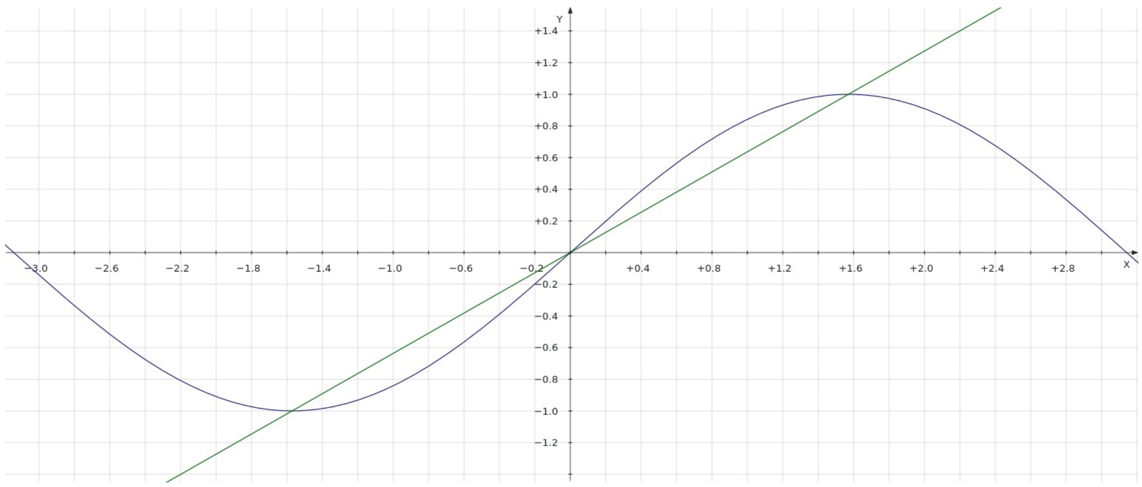
$$\begin{aligned}
A'(x) &= \frac{d}{dx}\left(8x\sqrt{1 - \frac{x^2}{9}}\right) = \left[\frac{d}{dx}(8x)\right]\sqrt{1 - \frac{x^2}{9}} + 8x\left[\frac{d}{dx}\sqrt{1 - \frac{x^2}{9}}\right] \\
&= 8\sqrt{1 - \frac{x^2}{9}} + 8x \cdot \frac{1}{2\sqrt{1 - \frac{x^2}{9}}} \cdot \frac{d}{dx}\left(1 - \frac{x^2}{9}\right) \\
&= 8\sqrt{1 - \frac{x^2}{9}} + \frac{8x}{2\sqrt{1 - \frac{x^2}{9}}} \cdot \left(-\frac{2x}{9}\right) = 8\sqrt{1 - \frac{x^2}{9}} - \frac{8x^2}{9\sqrt{1 - \frac{x^2}{9}}} \\
&= \frac{8 \cdot 9\left(\sqrt{1 - \frac{x^2}{9}}\right)^2 - 8x^2}{9\sqrt{1 - \frac{x^2}{9}}} = \frac{72\left(1 - \frac{x^2}{9}\right) - 8x^2}{9\sqrt{1 - \frac{x^2}{9}}} = \frac{72 - 8x^2 - 8x^2}{9\sqrt{1 - \frac{x^2}{9}}} \\
&= \frac{72 - 16x^2}{9\sqrt{1 - \frac{x^2}{9}}} = \frac{8\left(9 - 2x^2\right)}{9\sqrt{1 - \frac{x^2}{9}}} = 0 \Leftrightarrow 9 - 2x^2 = 0 \Leftrightarrow x = \pm\frac{3}{\sqrt{2}}
\end{aligned}$$

The critical point $x = -\frac{3}{\sqrt{2}} < 0$ so it is obviously not in the interval $[0, 3]$, but $x = \frac{3}{\sqrt{2}} \approx 2.1213$ is. Since $A\left(\frac{3}{\sqrt{2}}\right) = 8 \cdot \frac{3}{\sqrt{2}} \cdot \sqrt{1 - \frac{(3/\sqrt{2})^2}{9}} = 12\sqrt{2} \cdot \sqrt{1 - \frac{1}{2}} = 12\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 12$ is greater than $A(0) = A(3) = 0$, the maximum area of a rectangle which has each side parallel to one of the axes and all of its corners on the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is 12. ■

1. Sketch the finite region between $y = f(x)$ and $y = g(x)$ and find its area for:

b. $f(x) = \sin(x)$ and $g(x) = \dfrac{2x}{\pi}$. *[6]* **a.** $f(x) = \sin^2(x)$ and $g(x) = \dfrac{4x^2}{\pi^2}$. *[7]*

SOLUTIONS. **b.** To sort out the region, we first have to figure out where $f(x) = \sin(x)$ and $g(x) = \dfrac{2x}{\pi}$ intersect. It's easy to see that $f(0) = \sin(0) = 0$ and $g(0) = \dfrac{2 \cdot 0}{\pi} = 0$, so the graphs of the two functions cross at the origin. The $\dfrac{2}{\pi}$ in the definition of $g(x)$ is a clue to where else the curves might intersect: $g\left(\dfrac{\pi}{2}\right) = \dfrac{2}{\pi} \cdot \dfrac{\pi}{2} = 1$ and, not all coincidentally, $f\left(-\dfrac{\pi}{2}\right) = \sin\left(\dfrac{\pi}{2}\right) = 1$; similarly, $g\left(-\dfrac{\pi}{2}\right) = \dfrac{2}{\pi} \cdot \left(-\dfrac{\pi}{2}\right) = -1$ and $f\left(-\dfrac{\pi}{2}\right) = \sin\left(-\dfrac{\pi}{2}\right) =$ 1. A quick peek at the graph

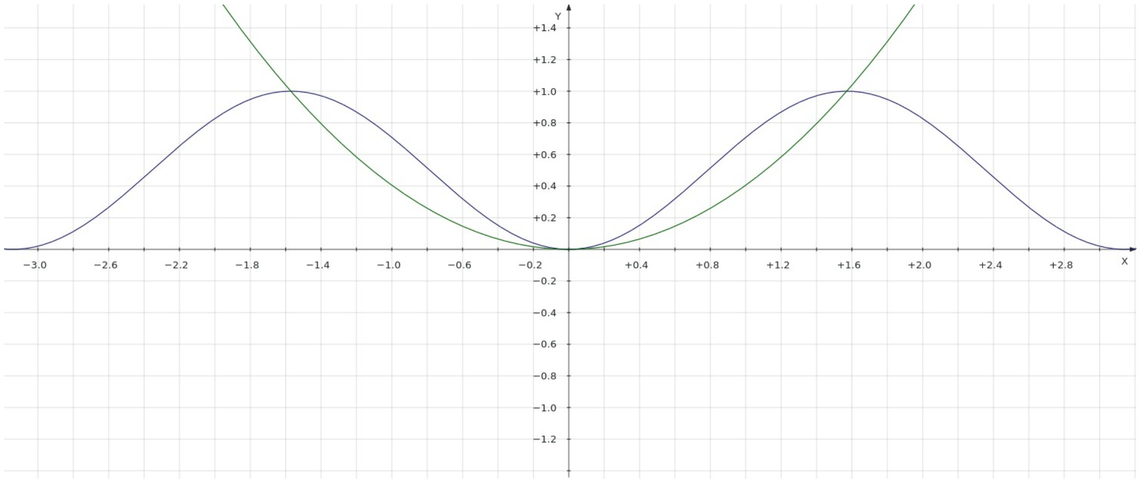


shows that these are the only points of intersection and indicates which of $f(x)$ and $g(x)$ is above the other between adjacent points of intersection. This can be tested by evaluating the functions between the points of intersection. One could also use the fact that the slope at $x = 0$ of $g(x)$ is $\frac{2}{\pi}$, which is less than the slope of $f(x)$ at $x = 0$, namely $f'(0) = \cos(0) = 1$. Note that the two parts of the region are symmetric about the origin – this follows from the fact that both functions are odd, *i.e.* $f(-x) = -f(x)$ and $g(-x) = -g(x)$ for all x – so each part has half the area, which fact saves us a bit of effort when integrating.

It follows that the area of the finite region between the two curves is:

$$\begin{aligned}
2\int_0^{\pi/2} [f(x) - g(x)]\, dx &= 2\int_0^{\pi/2} \left[\sin(x) - \frac{2x}{\pi}\right] dx = 2\left[-\cos(x) - \frac{2}{\pi} \cdot \frac{x^2}{2}\right]\Bigg|_0^{\pi/2} \\
&= 2\left[-\cos\left(\frac{\pi}{2}\right) - \frac{1}{\pi}\left(\frac{\pi}{2}\right)^2\right] - 2\left[-\cos(0) - \frac{1}{\pi}0^2\right] \\
&= 2\left[-0 - \frac{\pi}{4}\right] - 2\left[-1 - 0\right] = -\frac{\pi}{2} + 2 = 2 - \frac{\pi}{2} \quad \square
\end{aligned}$$

a. The $f(x)$ and $g(x)$ of this part are the squares of their counterparts in part **b**, so the x values that gave intersection pointsin part **b** still give intersection points in this part. A quick peek at the graph



again suggests these are the only points of intersection, and that the region comes in two parts that are symmetric about the y-axis. This last also follows from the fact that in this case both functions are even, *i.e.* $f(-x) = f(x)$ and $g(-x) = g(x)$ for all x. Again, one can determine which function is above the other by testing points between the intersection points.

It follows that the area of the finite region between the two curves is:

$$\begin{aligned}
2\int_0^{\pi/2} [f(x) - g(x)]\, dx &= 2\int_0^{\pi/2} \left[\sin^2(x) - \frac{4x^2}{\pi^2}\right] dx \\
&= 2\int_0^{\pi/2} \left[\left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right) - \frac{4x^2}{\pi^2}\right] dx \\
&= 2\left[\frac{x}{2} - \frac{1}{2}\cdot\frac{1}{2}\sin(2x) - \frac{4}{\pi^2}\cdot\frac{x^3}{3}\right]\Bigg|_0^{\pi/2} \\
&= \left[x - \frac{1}{2}\sin(2x) - \frac{8}{\pi^2}\cdot\frac{x^3}{3}\right]\Bigg|_0^{\pi/2} \\
&= \left[\frac{\pi}{2} - \frac{1}{2}\sin\left(2\frac{\pi}{2}\right) - \frac{8}{3\pi^2}\left(\frac{\pi}{2}\right)^3\right] - \left[0 - \frac{1}{2}\sin(2\cdot 0) - \frac{8\cdot 0^2}{\pi^2}\right] \\
&= \left[\frac{\pi}{2} - \frac{1}{2}\sin(\pi) - \frac{\pi}{3}\right] - \left[0 - \frac{1}{2}\cdot 0 - 0\right] \\
&= \frac{\pi}{2} - \frac{1}{2}\cdot 0 - \frac{\pi}{3} = \frac{\pi}{6} \quad \blacksquare
\end{aligned}$$

[Total = 100]

Part ○. Bonus problems! If you feel like it and have the time, do one or both of these.

◎. The longest straight line that fits inside a perfectly circular road of constant width is 100 m long. What is the area covered by the road? *[1]*

Solution. The area covered by the road is $2500\pi\ m^2$. You get to figure out why! :-) ■

⊙. Write a haiku touching on calculus or mathematics in general. *[1]*

What is a haiku?

seventeen in three:
five and seven and five of
syllables in lines

Solution. You're on your own here! ■

Enjoy the break!