Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Summer 2018

Assignment #2
Games With Limits
Due on Friday, 28 September.

The usual definition of limits,

> $\varepsilon - \delta$ DEFINITION OF LIMITS. $\lim_{x \to a} f(x) = L$ exactly when for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any x with $|x - a| < \delta$ we are guaranteed to have $|f(x) - L| < \varepsilon$ as well.

is pretty hard to wrap your head around the first time or three for most people. Here is less common definition, which is still equivalent to the one above, that is cast in terms of a game:

> LIMIT GAME DEFINITION OF LIMITS. The *limit game* for $f(x)$ at $x = a$ with target L is a three-move game played between two players A and B as follows:
>
> 1. A moves first, picking a small number $\varepsilon > 0$.
> 2. B moves second, picking another small number $\delta > 0$.
> 3. A moves third, picking an x that is within δ of a, *i.e.* $a - \delta < x < a + \delta$.
>
> To determine the winner, we evaluate $f(x)$. If it is within ε of the target L, *i.e.* $L - \varepsilon < f(x) < L + \varepsilon$, then player B wins; if not, then player A wins.
>
> With this idea in hand, $\lim_{x \to a} f(x) = L$ means that player B has a winning strategy in the limit game for $f(x)$ at $x = a$ with target L; that is, if B plays it right, B will win no matter what A tries to do. (Within the rules ... :-) Conversely, $\lim_{x \to a} f(x) \neq L$ means that player A is the one with a winning strategy in the limit game for $f(x)$ at $x = a$ with target L.

Your task in this assignment, should you choose to accept it, is to find such winning strategies:

1. Describe a winning strategy for B in the limit game for $f(x) = 2x + 3$ at $x = 2$ with target 7. Note that no matter what number ε A plays first, B must have a way to find a δ to play that will make it impossible for A to play an x that wins for A on the third move. *[3]*

2. Describe a winning strategy for A in the limit game for $f(x) = 2x + 3$ at $x = 2$ with target 8. Note that A must pick an ε on the first move so that no matter what δ B tries to play on the second move, A can still find an x to play on move three that wins for A. *[3]*

3. Use either definition of limits above to verify that $\lim_{x \to 2} \left(x^2 + 1\right) = 5$. *[4]*

Hint: The choice of δ in **3** will probably require some slightly indirect reasoning. Pick some arbitrary smallish positive number, say 1, for δ as a first cut. If it doesn't do the job, but x is at least that close, you'll have some more information to help pin down the δ you really need.

NOTE: The problems above are probably easiest done by hand, though `Maple` and its competitors do have tools for solving inequalities which could be useful.