Trent University

MATH 1101Y Test #2

~~Tuesday, 29~~ Wednesday, 30 January, 2013

Time: 50 minutes

Name: ______________________________

Student Number: ______________________________

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1	____
2	____
3	____
4	____
Total	____ /40

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Do any *three* (3) of **a**–**f**. *[12 = 3 × 4 each]*

$$\textbf{a.}\ \int \frac{1}{\sqrt{4-x^2}}\,dx \quad \textbf{b.}\ \int_{-1}^{1} (y+1)^2\,dy \quad \textbf{c.}\ \int \sec^2(w)\sqrt{\tan(w)}\,dw$$

$$\textbf{d.}\ \int_0^1 te^t\,dt \qquad \textbf{e.}\ \int \cos^3(x)\,dx \qquad \textbf{f.}\ \int_0^1 \frac{4}{1+x^2}\,dx$$

a. We will use the trig substitution $x = 2\sin(\theta)$, so $dx = 2\cos(\theta)\,d\theta$ and $\theta = \arcsin\left(\frac{x}{2}\right)$.

$$\begin{aligned}\int \frac{1}{\sqrt{4-x^2}}\,dx &= \int \frac{1}{\sqrt{4-2^2\sin^2(\theta)}}\,2\cos(\theta)\,d\theta = \int \frac{2\cos(\theta)}{\sqrt{4\left(1-\sin^2(\theta)\right)}}\,d\theta \\ &= \int \frac{2\cos(\theta)}{\sqrt{4\cos^2(\theta)}}\,d\theta = \int \frac{2\cos(\theta)}{2\cos(\theta)}\,d\theta \\ &= \int 1\,\theta = \theta + C = \arcsin\left(\frac{x}{2}\right) + C \qquad \blacksquare\end{aligned}$$

b. One could just expand the square and integrate away, but we'll do it using the substitution $u = y+1$, so $du = dy$ and $\begin{matrix} y & -1 & 1 \\ u & 0 & 2 \end{matrix}$. Easier arithmetic this way ...

$$\int_{-1}^{1} (y+1)^2\,dy = \int_0^2 u^2\,du = \left.\frac{u^3}{3}\right|_0^2 = \frac{8}{3} - \frac{0}{3} = \frac{8}{3} \qquad \blacksquare$$

c. We'll use the substitution $s = \tan(w)$, so $ds = \sec^2(w)\,dw$.

$$\begin{aligned}\int \sec^2(w)\sqrt{\tan(w)}\,dw &= \int \sqrt{s}\,ds = \int s^{1/2}\,ds = \frac{s^{3/2}}{3/2} + C \\ &= \frac{2}{3}s^{3/2} + C = \frac{2}{3}\tan^{3/2}(w) + C \qquad \blacksquare\end{aligned}$$

d. We will ise integration by parts, with $u = t$ and $v' = e^t$, so $u' = 1$ and $v = e^t$.

$$\begin{aligned}\int_0^1 te^t\,dt &= \int_0^1 uv'\,dt = uv\big|_0^1 - \int_0^1 u'v\,dt = te^t\big|_0^1 - \int_0^1 1e^t\,dt \\ &= \left(1e^1 - 0e^0\right) - e^t\big|_0^1 = e - \left(e^1 - e^0\right) = e - e + 1 = 1 \qquad \blacksquare\end{aligned}$$

e. This can be done with the help of the reduction formula for $\int \cos^k(x)\,dx$ or by using integration by parts, but the cheapest way is probably to use the trigonometric identity $\cos^2(x) = 1 - \sin^2(x)$ and the substitution $u = \sin(x)$, so $du = \cos(x)\,dx$.

$$\begin{aligned}\int \cos^3(x)\,dx &= \int \cos^2(x)\cos(x)\,dx = \int \left(1-\sin^2(x)\right)\cos(x)\,dx \\ &= \int \left(1-u^2\right)\,du = u - \frac{u^3}{3} + C = \sin(x) - \frac{1}{3}\sin^3(x) + C \qquad \blacksquare\end{aligned}$$

f. We'll use the fact that $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$ in reverse.

$$\int_0^1 \frac{4}{1+x^2}\,dx = 4\arctan(x)\big|_0^1 = 4\arctan(1) - 4\arctan(0) = 4\cdot\frac{\pi}{4} - 4\cdot 0 = \pi \qquad \blacksquare$$

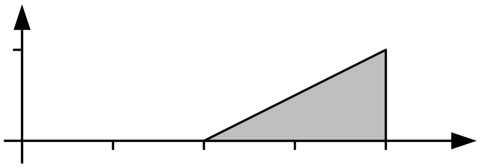
2. Do any *two* (2) of **a**–**c**. *[10 = 2 × 5 each]*

a. Sketch the region whose area is computed by the integral $\displaystyle\int_2^4 \left(\frac{x}{2}-1\right)\,dx$. Without evaluating the integral, what is its area?

b. Sketch the solid obtained by revolving the region below $y = 2$ and above $y = 1$, for $0 \le x \le 1$, about the x-axis, and find its volume.

c. Compute $\displaystyle\int_{-41\pi}^{41\pi} \arctan(\theta)\,d\theta$.

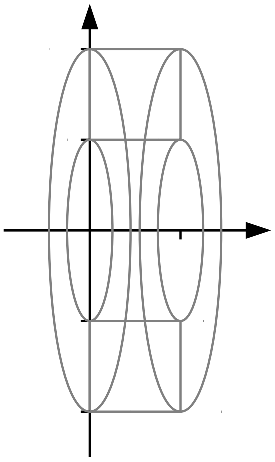
a. Here's a sketch of the region:



Note that the line $y = \frac{x}{2} - 1$ has x-intercept $x = 2$ and has $y = 1$ at $x = 4$, so it is above the x-axis for $2 < x \le 4$. $\displaystyle\int_2^4 \left(\frac{x}{2}-1\right)\,dx$ represents the area beneath this line and above the x-axis for $2 \le x \le 4$, but this region is just a triangle with base 2 and height 1. Thus

$$\int_2^4 \left(\frac{x}{2}-1\right)\,dx = \frac{1}{2}\cdot 2 \cdot 1 = 1\,. \qquad \square$$

b. Here's a sketch of the solid:



The volume is pretty easy to compute using either the washer or the shell method, but it's even faster if you remember that the volume of a cylinder of radius r and height h is $\pi r^2 h$, and notice that this solid is a cylinder of radius 2 and height 1 with a cylinder of radius 1 and height 1 removed from it. It follows that the volume of the solid is $V = \pi 2^2 \cdot 1 - \pi 1^2 \cdot 1 = 4\pi - \pi = 3\pi$. ■

c. $\arctan(\theta)$ is an odd function, that is, $\arctan(-\theta) = -\arctan(\theta)$. It follows that

$$\int_{-41\pi}^{0} \arctan(\theta)\, d\theta = -\int_{0}^{41\pi} \arctan(\theta)\, d\theta\,,$$

so

$$\int_{-41\pi}^{41\pi} \arctan(\theta)\, d\theta = \int_{-41\pi}^{0} \arctan(\theta)\, d\theta + \int_{0}^{41\pi} \arctan(\theta)\, d\theta = 0\,. \qquad \blacksquare$$

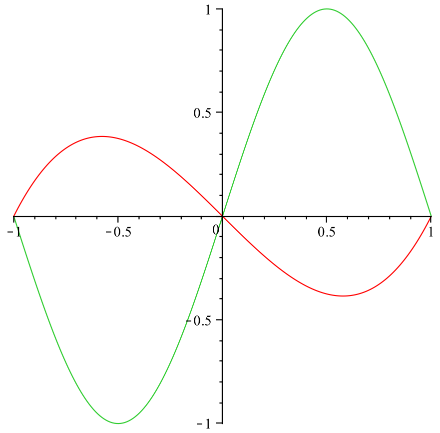
3. Do *one* (1) of **a** or **b**. *[8]*

a. Sketch the region between the curves $y = x^3 - x$ and $y = \sin(\pi x)$, where $-1 \le x \le 1$, and find its area.

b. Sketch the solid obtained by revolving the region between $y = \frac{1}{x}$ and $y = 1$, where $1 \le x \le 3$, about the line $x = -1$, and find its volume.

a. Cheating slightly, I used `Maple` to plot the curves:

```
> plot([[t,t^3-t,t=-1..1],[t,sin(Pi*t),t=-1..1]])
```



It's pretty clear from the plot that the curves intersect at $x = -1$, $x = 0$, and $x = 1$; for $-1 < x < 0$, $y = x^3 - x$ is above $y = \sin(\pi x)$, and for $0 < x < 1$, $y = \sin(\pi x)$ is above $y = x^3 - x$. (You will find all this out pretty quickly if you try to graph the curves by hand – it's not an accident the two curves intersect at their x-intercepts ... :-) It follows that the area of the region is:

$$A = \int_{-1}^{0} \left[\left(x^3 - x\right) - \sin(\pi x)\right]\,dx + \int_{0}^{1} \left[\sin(\pi x) - \left(x^3 - x\right)\right]\,dx$$

$$= \int_{-1}^{0} \left(x^3 - x\right)\,dx - \int_{-1}^{0} \sin(\pi x)\,dx + \int_{0}^{1} \sin(\pi x)\,dx - \int_{0}^{1} \left(x^3 - x\right)\,dx$$

We'll substitute $u = \pi x$, so $du = \pi\,dx$ and $\frac{1}{\pi}\,du = dx$, in the middle integrals.

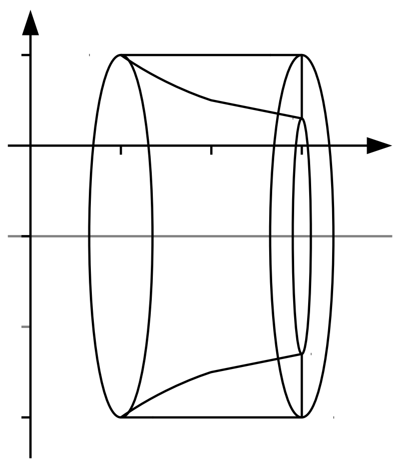
$$= \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Big|_{-1}^{0} - \int_{-\pi}^{0} \sin(u)\,\frac{1}{\pi}\,du + \int_{0}^{\pi} \sin(u)\,\frac{1}{\pi}\,du - \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Big|_{0}^{1}$$

$$= 0 - \left(-\frac{1}{4}\right) - \frac{1}{\pi}\left(-\cos(u)\right)\Big|_{-\pi}^{0} + \frac{1}{\pi}\left(-\cos(u)\right)\Big|_{0}^{\pi} - \left(-\frac{1}{4}\right) + 0$$

$$= \frac{1}{4} + \frac{1}{\pi}\left(\cos(0) - \cos(-\pi)\right) - \frac{1}{\pi}\left(\cos(\pi) - \cos(0)\right) + \frac{1}{4}$$

$$= \frac{1}{2} + \frac{1}{\pi}\left(1 - (-1)\right) - \frac{1}{\pi}\left((-1) - 1\right) = \frac{1}{2} + \frac{4}{\pi} \qquad \blacksquare$$

b. Here's a sketch of the solid obtained if one revolves the region about the line $x = -1$:



The volume of this solid is pretty easy to compute using either the washer or the cylindrical shell method. We'll use washers:

Since we revolved about a horizontal line and intend to use washers, we will use x as the basic variable. Observe that the washer at x has outer radius $R = 1 - (-1) = 2$ and inner radius $r = \frac{1}{x} - (-1) = \frac{1}{x} + 1$, so its area is

$$A(x) = \pi R^2 - \pi r^2 = \pi\left(2^2 - \left(\frac{1}{x} + 1\right)^2\right) = \pi\left(4 - \left(\frac{1}{x^2} - \frac{2}{x} + 1\right)\right) = \pi\left(3 + \frac{2}{x} - \frac{1}{x^2}\right).$$

The volume of the solid is therefore

$$\begin{aligned}
V &= \int_1^3 A(x)\,dx = \int_1^3 \pi\left(3 + \frac{2}{x} - \frac{1}{x^2}\right)\,dx = \pi\int_1^3 \left(3 + 2x^{-1} - x^{-2}\right)\,dx \\
&= \pi\left(3x + 2\ln(x) - \frac{x^{-1}}{-1}\right)\Bigg|_1^3 = \pi\left(3x + 2\ln(x) + \frac{1}{x}\right)\Bigg|_1^3 \\
&= \pi\left(3\cdot 3 + 2\ln(3) + \frac{1}{3}\right) - \pi\left(3\cdot 1 + 2\ln(1) + \frac{1}{1}\right) \\
&= \pi\left(9 + 2\ln(3) + \frac{1}{3}\right) - \pi\,(4 + 2\cdot 0) \\
&= \pi\left(5 + \frac{1}{3} + 2\ln(3)\right) = \pi\left(\frac{16}{3} + 2\ln(3)\right) \qquad \blacksquare
\end{aligned}$$

4. Do *one* (1) of **a** or **b**. *[10]*

a. Find the domain and any and all intercepts, horizontal and vertical asymptotes, local maxima and minima, and inflection points of $f(x) = e^{-x^2}$, and sketch its graph.

b. Max moves at 1 km/hr along the positive x-axis towards the origin while aiming a laser pointer at the $(0,2)$ on the y-axis. How is the (smaller!) angle between the laser beam and the the x-axis changing at the instant that Max is at the point $(1,0)$ on the x-axis? (All distances along the axes are in kilometres. You may assume Max and the laser pointer occupy a single point at any given instant ... :-)

a. We run through the usual checklist:

i. Domain. $f(x) = e^{-x^2}$ makes sense for all x, so the domain is the entire real line. Note that since $f(x)$ is a composition of functions which are everywhere continuous, it is also continuous everywhere. $\square$

ii. Intercepts. $f(0) = e^{-0^2} = e^0 = 1$ so $f(x) = e^{-x^2}$ has y-intercept 1. On the other hand, $e^t > 0$ for every $t \in \mathbb{R}$, so $f(x) = e^{-x^2} > 0$ for all x, and so there are no x-intercepts. $\square$

iii. Vertical asymptotes. Since $f(x) = e^{-x^2}$ is defined and continuous for all x, it has no vertical asymptotes. $\square$

iv. Horizontal asymptotes. We compute the limits in both directions:

$$\lim_{x \to -\infty} e^{-x^2} = 0 \quad \text{since } -x^2 \to -\infty \text{ as } x \to -\infty \text{ and } e^t \to 0 \text{ as } t \to -\infty \text{, and}$$
$$\lim_{x \to +\infty} e^{-x^2} = 0 \quad \text{since } -x^2 \to -\infty \text{ as } x \to +\infty \text{ and } e^t \to 0 \text{ as } t \to -\infty.$$

Thus $f(x) = e^{-x^2}$ has the horizontal asymptote $y = 0$ in both directions. $\square$

v. Maxima, minima, etc. First, $f'(x) = \frac{d}{dx} e^{-x^2} = e^{-x^2} \frac{d}{dx}\left(-x^2\right) = -2xe^{-x^2}$. Since, as observed in *ii* above, $e^{-x^2} > 0$ for all x, $f'(x) = -2xe^{-x^2} = 0$ exactly when $x = 0$. Note that it also follows that if $x < 0$, $f'(x) > 0$, and that if $x > 0$, $f'(x) < 0$. The usual table then amounts to:

x	$(-\infty, 0)$	0	$(0, \infty)$
$f'(x)$	$+$	0	$-$
$f(x)$	$\uparrow$	max	$\downarrow$

Thus $f(x)$ has a local maximum (of $f(0) = 1$) at $x = 0$ and has no local minimum. (A little reflection about the table above should convince you that this local maximum is also an absolute maximum of $f(x)$.) $\square$

vi. Curvature and inflection points. First,

$$\begin{aligned} f''(x) &= \frac{d}{dx} f'(x) = \frac{d}{dx}\left(-2xe^{-x^2}\right) = -2\left(\frac{d}{dx}x\right)e^{-x^2} + (-2x)\left(\frac{d}{dx}e^{-x^2}\right) \\ &= -2e^{-x^2} + (-2x)\left(-2xe^{-x^2}\right) = \left(4x^2 - 2\right)e^{-x^2}. \end{aligned}$$

Since, as observed in *ii* above, $e^{-x^2} > 0$ for all x, $f''(x) = \left(4x^2 - 2\right)e^{-x^2} = 0$ exactly when $4x^2 - 2 = 0$, *i.e.* when $x = \pm\frac{1}{\sqrt{2}}$. Note that it also follows that if $x^2 < \frac{1}{2}$, *i.e.*

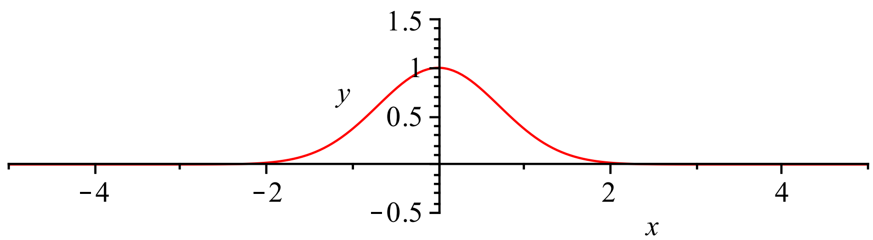
$-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$, then $f''(x) < 0$, and that if $x^2 > \frac{1}{2}$, *i.e.* $\left|\frac{1}{\sqrt{2}}\right| > 0$, then $f''(x) > 0$. The usual table then amounts to:

x	$\left(-\infty, -\frac{1}{\sqrt{2}}\right)$	$-\frac{1}{\sqrt{2}}$	$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\frac{1}{\sqrt{2}}$	$\left(\frac{1}{\sqrt{2}}, \infty\right)$
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\smile$	inflection point	$\frown$	inflection point	$\smile$

Thus $f(x) = e^{-x^2}$ has two inflection points, at $x = -\frac{1}{\sqrt{2}}$ and $x = \frac{1}{\sqrt{2}}$. It is concave down between them and concave up to either side. $\square$

vii. The graph. Cheating slightly, here is what `Maple` gives:

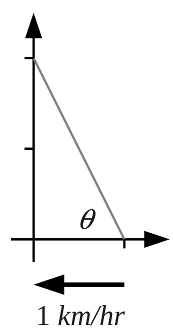
```
> plot(exp(-x^2),x=-5..5,y=-0.5..1.5)
```



$\square$

All done! $\blacksquare$

b. A crude sketch of the set-up is on the right.



If x is Max's position on the x-axis at some instant, then $\frac{dx}{dt} = -1\ km/hr$ because Max is moving towards the origin from the right. The corresponding θ satisfies $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{2}{x}$, so $\theta = \arctan\left(\frac{2}{x}\right)$. It follows that at every instant

$$\frac{d\theta}{dt} = \frac{d}{dt}\arctan\left(\frac{2}{x}\right) = \left[\frac{d}{dx}\arctan\left(\frac{2}{x}\right)\right] \cdot \frac{dx}{dt} = \left[\frac{1}{1+\left(\frac{2}{x}\right)^2} \cdot \frac{d}{dx}\left(\frac{2}{x}\right)\right] \cdot (-1)$$
$$= -\left[\frac{1}{1+\frac{4}{x^2}} \cdot \left(-\frac{2}{x^2}\right)\right] = \frac{2}{x^2+4}\,.$$

Thus, when $x = 1$, $\frac{d\theta}{dt} = \frac{2}{1^2+4} = \frac{2}{5} = 0.4\ rad/hr$, that is, the angle between the laser beam and the the x-axis is increasing at the rate of 0.4 radians per hour at the instant that Max is at $(0, 1)$ on the x-axis. [Why radians per hour?] $\blacksquare$

[Total = 40]