Mathematics 1101Y – Calculus I: functions and calculus of one variable
TRENT UNIVERSITY, 2012–2013
Solutions to the Final Examination

Time: *09:00–12:00, on Thursday, 11 April, 2013.* *Brought to you by* Стефан Біланюк.

Instructions: Do parts **I**, **J**, and **K**, and, if you wish, part **Z**. Show all your work and justify all your answers. *If in doubt about something,* **ask!**

Aids: Any calculator; (all sides of) one aid sheet; one (1) brain ($10^{10^{10}}$ neuron limit).

Part I. Do all four (4) of **1**–**4**.

1. Compute $\frac{dy}{dx}$ as best you can in any *three* (3) of **a**–**f**. *[15 = 3 × 5 each]*

a. $y = \dfrac{e^{2x}-1}{e^{2x}+1}$ **b.** $\begin{array}{l} y = \arctan(t) \\ x = \frac{1}{3}t^3 + t \end{array}$ **c.** $y = (1+\sin(x))^2$

d. $\tan(y) = x$ **e.** $y = xe^{-x}$ **f.** $y = \displaystyle\int_1^x \frac{\ln(t)}{t}\,dt$

SOLUTIONS. **a.** We'll use the Quotient and Chain Rules:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{e^{2x}-1}{e^{2x}+1}\right) = \frac{\left[\frac{d}{dx}\left(e^{2x}-1\right)\right]\left(e^{2x}+1\right)-\left(e^{2x}-1\right)\left[\frac{d}{dx}\left(e^{2x}+1\right)\right]}{\left(e^{2x}+1\right)^2} \\ &= \frac{\left[e^{2x}\frac{d}{dx}(2x)-0\right]\left(e^{2x}+1\right)-\left(e^{2x}-1\right)\left[e^{2x}\frac{d}{dx}(2x)+0\right]}{\left(e^{2x}+1\right)^2} \\ &= \frac{2e^{2x}\left(e^{2x}+1\right)-\left(e^{2x}-1\right)2e^{2x}}{\left(e^{2x}+1\right)^2} = \frac{4e^{2x}}{\left(e^{2x}+1\right)^2} \qquad \blacksquare\end{aligned}$$

b. $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\arctan(t)}{\frac{d}{dt}\left(\frac{1}{3}t^3+t\right)} = \frac{\frac{1}{1+t^2}}{\frac{1}{3}\cdot 3t^2+1} = \frac{1}{\left(1+t^2\right)^2} \qquad \blacksquare$$

c. We'll use the Power and Chain Rules:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(1+\sin(x))^2 = 2(1+\sin(x))\cdot\frac{d}{dx}(1+\sin(x)) \\ &= 2(1+\sin(x))\cdot(0+\cos(x)) = 2\cos(x)(1+\sin(x)) \qquad \blacksquare\end{aligned}$$

d. $\tan(y) = x \implies y = \arctan(x)$, so $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$. $\blacksquare$

e. We'll use the Product and Chain Rules:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(xe^{-x}\right) = \left(\frac{d}{dx}x\right)e^{-x} + x\left(\frac{d}{dx}e^{-x}\right) \\ &= 1e^{-x} + xe^{-x}\frac{d}{dx}(-x) = e^{-x} + xe^{-x}(-1) = (1-x)e^{-x} \qquad \blacksquare\end{aligned}$$

f. This is a job for the Fundamental Theorem of Calculus:

$$\frac{dy}{dx} = \frac{d}{dx}\int_1^x \frac{\ln(t)}{t}\,dt = \frac{\ln(x)}{x} \qquad \blacksquare$$

2. Evaluate any *three* (3) of the integrals **a**–**f**. *[15 = 3 × 5 each]*

a. $\displaystyle\int \sec^{17}(x)\tan(x)\,dx$ **b.** $\displaystyle\int_0^{\sqrt{\pi}} z\cos\left(z^2\right)\,dz$ **c.** $\displaystyle\int \frac{1}{\sqrt{4+x^2}}\,dx$

d. $\displaystyle\int_0^1 \arctan(y)\,dy$ **e.** $\displaystyle\int \frac{1}{x^3+x}\,dx$ **f.** $\displaystyle\int_1^\infty \frac{1}{t^2}\,dt$

SOLUTIONS. **a.** We'll use the substitution $u = \sec(x)$, so $du = \sec(x)\tan(x)\,dx$.

$$\begin{aligned}\int \sec^{17}(x)\tan(x)\,dx &= \int \sec^{16}(x)\sec(x)\tan(x)\,dx = \int u^{16}\,du \\ &= \frac{1}{17}u^{17} + C = \frac{1}{17}\sec^{17}(x) + C \qquad \blacksquare\end{aligned}$$

b. We'll use the substitution $w = z^2$, so $dw = 2z\,dz$ and thus $z\,dz = \frac{1}{2}\,dw$, and $\begin{array}{ccc} z & 0 & \sqrt{\pi} \\ w & 0 & \pi \end{array}$.

$$\int_0^{\sqrt{\pi}} z\cos\left(z^2\right)\,dz = \int_0^{\pi} \cos(w)\cdot\frac{1}{2}\,dw = \frac{1}{2}\sin(w)\Big|_0^{\pi} = \frac{1}{2}\sin(\pi) - \frac{1}{2}\sin(0) = 0 - 0 = 0 \qquad \blacksquare$$

c. We'll use the trigonometric substitution $x = 2\tan(\theta)$, so $dx = 2\sec^2(\theta)\,d\theta$. Note that $\tan(\theta) = \frac{x}{2}$ and $\sec(\theta) = \sqrt{1+\tan^2(\theta)} = \sqrt{1+\frac{x^2}{4}}$.

$$\begin{aligned}\int \frac{1}{\sqrt{4+x^2}}\,dx &= \int \frac{1}{\sqrt{4+4\tan^2(\theta)}}\,2\sec^2(\theta)\,d\theta = \int \frac{2\sec^2(\theta)}{\sqrt{4\left(1+\tan^2(\theta)\right)}}\,d\theta \\ &= \int \frac{2\sec^2(\theta)}{\sqrt{4\sec^2(\theta)}}\,d\theta = \int \frac{2\sec^2(\theta)}{2\sec(\theta)}\,d\theta = \int \sec(\theta)\,d\theta \\ &= \ln\left(\sec(\theta)+\tan(\theta)\right) + C = \ln\left(\sqrt{1+\frac{x^2}{4}} + \frac{x}{2}\right) + C \qquad \blacksquare\end{aligned}$$

d. We'll use integration by parts with $u = \arctan(y)$ and $v' = 1$, so $u' = \frac{1}{1+y^2}$ and $v = y$. The remaining integral will be done using the substitution $w = 1+y^2$, so $dw = 2y\,dy$, and thus $y\,dy = \frac{1}{2}\,dw$, and $\begin{array}{ccc} y & 0 & 1 \\ w & 1 & 2 \end{array}$.

$$\begin{aligned}\int_0^1 \arctan(y)\,dy &= \int_0^1 uv'\,dy = uv|_0^1 - \int_0^1 u'v\,dy = y\arctan(y)|_0^1 - \int_0^1 \frac{y}{1+y^2}\,dy \\ &= [1\arctan(1) - 0\arctan(0)] - \int_1^2 \frac{1}{w}\frac{1}{2}\,dw = \left[\frac{\pi}{4} - 0\right] - \frac{1}{2}\ln\left(\frac{1}{w}\right)\Big|_1^2 \\ &= \frac{\pi}{4} - \left[\frac{1}{2}\ln\left(\frac{1}{2}\right) - \frac{1}{2}\ln\left(\frac{1}{1}\right)\right] = \frac{\pi}{4} - \frac{1}{2}\ln\left(\frac{1}{2}\right) \qquad \blacksquare\end{aligned}$$

e. $\dfrac{1}{x^3+x}$ is a rational function with degree of the denominator, 3, greater than the degree of the numerator, 0. Since $x^3+x = x\left(x^2+1\right)$, where x^2+1 is an irreducible quadratic (because $x^2+1 \geq 0+1 = 1 > 0$ for all x), we get that

$$\frac{1}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A\left(x^2+1\right)+(Bx+C)x}{x\left(x^2+1\right)} = \frac{(A+B)x^2+Cx+A}{x^3+x}$$

for some unknown constants A, B, and C. Comparing coefficients in the numerators, this requires that $A+B=0$, $C=0$, and $A=1$, so $B=-1$. It follows that

$$\begin{aligned}\int \frac{1}{x^3+x}\,dx &= \int \left(\frac{1}{x} + \frac{-x}{x^2+1}\right) dx = \int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx = \ln(x) - \int \frac{1}{u}\cdot\frac{1}{2}\,du \\ &= \ln(x) - \frac{1}{2}\ln(u) + C = \ln(x) - \frac{1}{2}\ln\left(x^2+1\right) + C\,,\end{aligned}$$

where we used the substitution $u = x^2+1$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. ■

f. We will use the Power Rule along the way:

$$\begin{aligned}\int_1^\infty \frac{1}{t^2}\,dt &= \lim_{z\to\infty}\int_1^z \frac{1}{t^2}\,dt = \lim_{z\to\infty}\int_1^z t^{-2}\,dt = \lim_{z\to\infty} -t^{-1}\Big|_1^z = \lim_{z\to\infty} -\frac{1}{t}\Big|_1^z \\ &= \lim_{z\to\infty}\left[-\frac{1}{z} - \left(-\frac{1}{1}\right)\right] = \lim_{z\to\infty}\left[1-\frac{1}{z}\right] = 1-0 = 1 \qquad ■\end{aligned}$$

3. Do any *three* (3) of **a**–**f**. *[15 = 3 × 5 each]*

- **a.** Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n^2}x^n$.
- **b.** Sketch the polar curve $r=\theta$, $0 \leq \theta \leq \pi$, and find the area of the region between this curve and the origin.
- **c.** Determine whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{\sqrt{n}}{(n+1)^2}$ converges or diverges.
- **d.** Sketch the region between $y=x^2$ and $y=\sqrt{x}$, $0 \leq x \leq 1$, and find its area.
- **e.** Sketch the parametric curve $x=\cos(t)$, $y=\sin(t)$, $0 \leq x \leq \pi$, and find its arc-length.
- **f.** Compute $f'(0)$ using the limit definition of the derivative if $f(x)=x^2+x+1$.
- **g.** Sketch the solid obtained by revolving the region between $y=1$ and $y=\sqrt{x}$, $0 \leq x \leq 1$, about the y-axis, and find its volume.

SOLUTIONS. **a.** Ahoy, good ship Ratio Test!

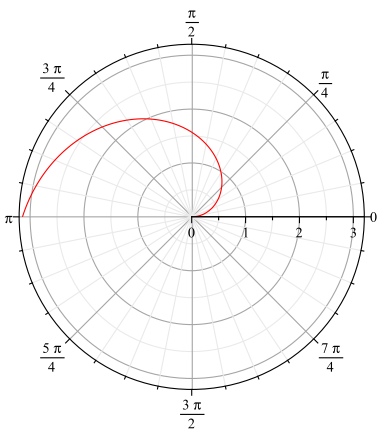
$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{(n+1)^2}x^{n+1}}{\frac{2^n}{n^2}x^n}\right| = \lim_{n\to\infty}\frac{n^2}{(n+1)^2}2|x| = 2|x|\lim_{n\to\infty}\frac{n^2}{n^2+2n+1}\cdot\frac{1/n^2}{1/n^2} \\ &= 2|x|\lim_{n\to\infty}\frac{1}{1+\frac{2}{n}+\frac{1}{n^2}} = 2|x|\lim_{n\to\infty}\frac{1}{1+0+0} = 2|x|\end{aligned}$$

It follows by the Ratio Test that the series converges when $2|x| < 1$, that is, when $|x| < \frac{1}{2}$, and diverges when $2|x| > 1$, that is, when $|x| > \frac{1}{2}$. Hence the radius of convergence of the given power series is $R = \frac{1}{2}$. ■

b. Here's is the curve, as plotted by `Maple`:

$>$ `with(plots):`

$>$ `polarplot(t,t=0..Pi)`



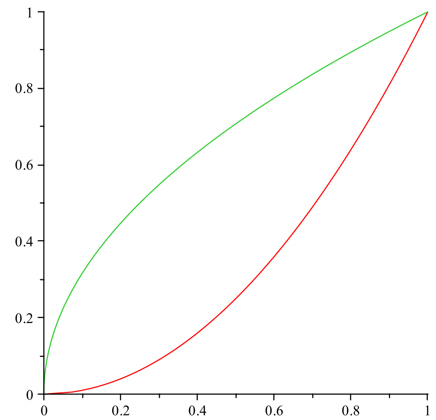
To find the area of the region between the curve and the origin, we use the usual area formula for polar regions:

$$\text{Area} = \int_0^\pi \frac{1}{2} r^2 \, d\theta = \int_0^\pi \frac{1}{2} \theta^2 \, d\theta = \frac{1}{2} \cdot \frac{\theta^3}{3} \Big|_0^\pi = \frac{\pi^3}{6} - \frac{0^3}{6} = \frac{\pi^3}{6} \qquad \blacksquare$$

c. $\displaystyle\sum_{n=0}^{\infty} \frac{\sqrt{n}}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{n^{1/2}}{n^2 + 2n + 1}$ is a series whose terms are given by a rational function of n, albeit with a fractional exponent in the numerator. The difference between the degree of the denominator and the degree of the numerator is $p = 2 - \frac{1}{2} = \frac{3}{2} > 1$, so the series converges by the Generalized p-Test. ■

d. Here's is the curve, as plotted by `Maple`:

> `plot([[sqrt(t),t,t=0..1],[t^2,t,t=0..1]]s)`



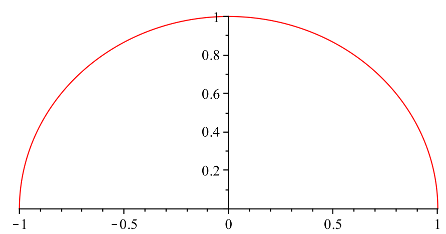
The two curves intersect at $x = 0$ and $x = 1$; between these two points, $\sqrt{x} \geq x^2$. It follows that the area between the curves is given by:

$$\text{Area} = \int_0^1 \left(\sqrt{x} - x^2\right) dx = \int_0^1 \left(x^{1/2} - x^2\right) dx = \left(\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right)\Bigg|_0^1$$

$$= \left(\frac{2}{3}1^{3/2} - \frac{1}{3}1^3\right) - \left(\frac{2}{3}0^{3/2} - \frac{1}{3}0^3\right) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \qquad \blacksquare$$

e. Here's what `Maple` thinks the curve looks like:

> `plot([cos(t),sin(t),t=0..Pi])`



(Since $x^2 + y^2 = \cos^(t) + \sin^2(t) = 1$ for any point on this curve, it is a piece of the unit circle ...)

To find the length of the curve, we plug its definition into the variant of the arc-length formula for parametric curves:

$$\text{arc-length} = \int_0^\pi ds = \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^\pi \sqrt{\left(\frac{d}{dt}\cos(t)\right)^2 + \left(\frac{d}{dt}\sin(t)\right)^2}\, dt$$

$$= \int_0^\pi \sqrt{(-\sin(t))^2 + (\cos(t))^2}\, dt = \int_0^\pi \sqrt{\sin^2(t) + \cos^2(t)}\, dt = \int_0^\pi \sqrt{1}\, dt$$

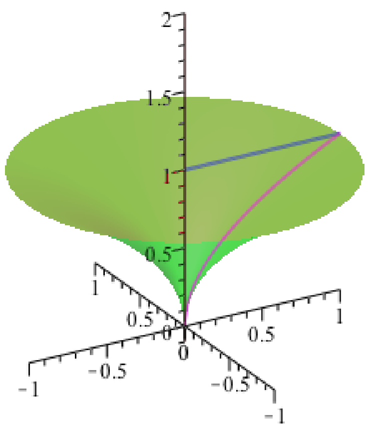
$$= \int_0^\pi 1\, dt = t\big|_0^\pi = \pi - 0 = \pi \qquad \blacksquare$$

f. Here goes:

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{\left(h^2+h+1\right) - \left(0^2+0+1\right)}{h}$$
$$= \lim_{h \to 0} \frac{h^2+h}{h} = \lim_{h \to 0} (h+1) = 0+1 = 1 \qquad \blacksquare$$

g. Here is `Maple`'s depiction of the solid in question:

```
> with(Student[Calculus1]):
> VolumeOfRevolution(1,sqrt(x),x=0..1,axis=vertical,output=plot,
  transparencv=0.75.title=" ")
```



We'll use the disk/washer method to compute the volume of the solid. Since the axis of revolution was the y-axis, the disks are stacked vertically and we will need to use y as the basic variable. Note first that $0 \le y \le 1$ over the given region. For any given y in this range, the disk in question has radius $R = x - 0 = x$, where $y = \sqrt{x}$, so $R = x = y^2$ in terms of y. (Note also that each disk has no hole here ...) It follows that the volume of the region is given by:

$$V = \int_0^1 \pi R^2 \, dy = \int_0^1 \pi \left(y^2\right)^2 \, dy = \pi \int_0^1 y^4 \, dy = \pi \frac{y^5}{5}\bigg|_0^1 = \pi \frac{1^5}{5} - \pi \frac{0^5}{5} = \frac{\pi}{5} \qquad \blacksquare$$

4. Consider the curve $y = \dfrac{x^2}{2}$, for $0 \leq x \leq 2$.

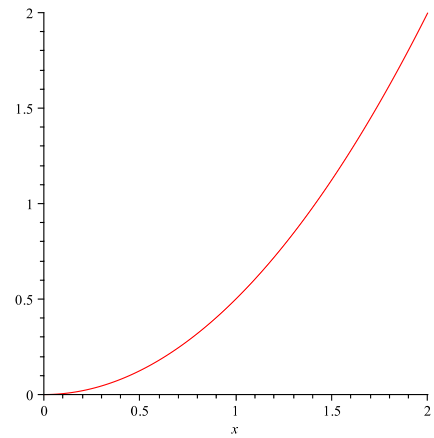
a. Sketch the curve. *[1]*

b. Sketch the surface obtained by revolving the curve about the x-axis. *[1]*

c. Compute either *i.* the length of the curve or *ii.* the area of the surface. [Just one, please!] *[8]*

SOLUTIONS. **a.** `Maple` strikes again:

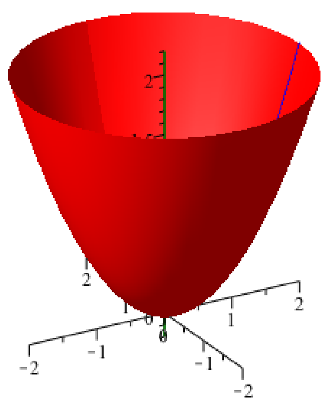
```
> plot(x^2/2,x=0..2
```



■

b. ... and again:

```
> with(Student[Calculus1]):
> SurfaceOfRevolution(x^2/2,x=0..2,axis=vertical,output=plot,title=" ")
```



■

Oops! Wrong axis ...

c. *i.* $\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{x^2}{2} = \dfrac{2x}{2} = x$, so $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+x^2}\,dx$. Hence, using the substitution $x = \tan(\theta)$, so $dx = \sec^2(\theta)$ and $\sqrt{1+x^2} = \sqrt{1+\tan^2(\theta)} = \sqrt{\sec^2(\theta)} = \sec(\theta)$:

$$\begin{aligned}
\text{arc-length} &= \int_0^2 ds = \int_0^2 \sqrt{1+x^2}\,dx = \int_{x=0}^{x=2} \sec(\theta)\,\sec^2(\theta)\,d\theta = \int_{x=0}^{x=2} \sec^3(\theta)\,d\theta \\
&= \left[\frac{1}{2}\sec(\theta)\tan(\theta) + \frac{1}{2}\ln\left(\sec(\theta)+\tan(\theta)\right)\right]\Bigg|_{x=0}^{x=2} \\
&= \left[\frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln\left(x+\sqrt{1+x^2}\right)\right]\Bigg|_0^2 \\
&= \left[\frac{1}{2}2\sqrt{1+2^2} + \frac{1}{2}\ln\left(2+\sqrt{1+2^2}\right)\right] - \left[\frac{1}{2}0\sqrt{1+0^2} + \frac{1}{2}\ln\left(0+\sqrt{1+0^2}\right)\right] \\
&= \sqrt{5} + \frac{1}{2}\ln\left(2+\sqrt{5}\right) - 0 - \frac{1}{2}\ln(1) = \sqrt{5} + \frac{1}{2}\ln\left(2+\sqrt{5}\right) \qquad \blacksquare
\end{aligned}$$

c. *ii.* $\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{x^2}{2} = \dfrac{2x}{2} = x$, so $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+x^2}\,dx$. Also, since we are revolving the curve about the x-axis, the point at x on the curve is revolved around a circle with radius $r = y - 0 = \frac{x^2}{2}$. Hence, using the trigonometric substitution $x = \tan(\theta)$, so $dx = \sec^2(\theta)$ and $\sqrt{1+x^2} = \sqrt{1+\tan^2(\theta)} = \sqrt{\sec^2(\theta)} = \sec(\theta)$::

$$\begin{aligned}
\text{surface area} &= \int_0^2 2\pi r\,ds = \int_0^2 2\pi\frac{x^2}{2}\sqrt{1+x^2}\,dx = \int_{x=0}^{x=2} \pi\frac{\tan^2(\theta)}{2}\sec(\theta)\,d\theta \\
&= \frac{\pi}{2}\int_{x=0}^{x=2} \tan^2(\theta)\sec(\theta)\,d\theta = \frac{\pi}{2}\int_{x=0}^{x=2} \left(\sec^2(\theta)-1\right)\sec(\theta)\,d\theta \\
&= \frac{\pi}{2}\int_{x=0}^{x=2} \sec^3(\theta)\,d\theta - \frac{\pi}{2}\int_{x=0}^{x=2} \sec(\theta)\,d\theta \\
&= \frac{\pi}{2}\left[\frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln\left(x+\sqrt{1+x^2}\right)\right]\Bigg|_0^2 - \frac{\pi}{2}\left[\ln\left(x+\sqrt{1+x^2}\right)\right]\Bigg|_0^2 \\
&= \frac{\pi}{2}\left[\frac{1}{2}x\sqrt{1+x^2} - \frac{1}{2}\ln\left(x+\sqrt{1+x^2}\right)\right]\Bigg|_0^2 \\
&= \frac{\pi}{2}\left[\frac{1}{2}2\sqrt{1+2^2} - \frac{1}{2}\ln\left(2+\sqrt{1+2^2}\right)\right] \\
&\qquad - \frac{\pi}{2}\left[\frac{1}{2}0\sqrt{1+0^2} - \frac{1}{2}\ln\left(0+\sqrt{1+0^2}\right)\right] \\
&= \frac{\pi}{2}\left[\sqrt{5} - \frac{1}{2}\ln\left(2+\sqrt{5}\right) - 0 + \frac{1}{2}\ln(1)\right] \\
&= \frac{\pi}{2}\left[\sqrt{5} - \frac{1}{2}\ln\left(2+\sqrt{5}\right)\right] \qquad \blacksquare
\end{aligned}$$

Part J. Do any *two* (2) of **5–7**. *[30 = 2 × 15 each]*

5. Gravel is dumped from a conveyor belt at a rate of 3 m^3/min. At any given instant the gravel forms a conical pile whose height is twice the radius of the base. How fast is the height of the pile increasing at the instant that the pile is 1 m high? [The volume of a cone with height h and base radius r is $\frac{1}{3}\pi r^2 h$.]

SOLUTION. Since the height of the conical pile is always twice the radius of the base, *i.e.* $h = 2r$ or $r = \frac{h}{2}$, the volume of the cone is given by $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}$. It follows that

$$3 = \frac{dV}{dt} = \frac{d}{dt}\frac{1}{12}\pi h^3 = \left(\frac{d}{dh}\frac{\pi h^3}{12}\right)\cdot\frac{dh}{dt} = \frac{\pi h^2}{4}\cdot\frac{dh}{dt},$$

so $\frac{dh}{dt} = \frac{12}{\pi h^2}$ at any given instant. Plugging in $h = 1\ m$ then gives

$$\left.\frac{dh}{dt}\right|_{h=1\ m} = \frac{12}{\pi 1^2} = \frac{12}{\pi}\ m/s. \qquad \blacksquare$$

6. Find any and all intercepts, maximum, minimum, and inflection points, and vertical and horizontal asymptotes of $f(x) = e^{1/x}$, and sketch its graph.

SOLUTION. We run through the usual checklist:

i. *(Domain)* Since $1/x$ is defined (and continuous and differentiable) for all $x \neq 0$ and e^t is defined (and continuous and differentiable) for all t, $f(x) = e^{1/x}$ is defined (and continuous and differentiable) for all $x \neq 0$. $\square$

ii. *(Intercepts)* $f(x) = e^{1/x}$ is not defined at $x = 0$, so there is no y-intercept. Since $e^t > 0$ for all t, $f(x) = e^{1/x} > 0$ for all $x \neq 0$, so there is no x-intercept either. $\square$

iii. *(Vertical asymptotes)* Since $f(x) = e^{1/x}$ is defined and continuous for all $x \neq 0$, the only place there might be a vertical asymptote is at $x = 0$. Let's check:

$$\lim_{x\to 0^+} e^{1/x} = +\infty \quad \text{since } \frac{1}{x} \to +\infty \text{ as } x \to 0^+ \text{ and } e^t \to +\infty \text{ as } t = 1/x \to +\infty$$
$$\lim_{x\to 0^-} e^{1/x} = 0 \quad \text{since } \frac{1}{x} \to -\infty \text{ as } x \to 0^- \text{ and } e^t \to 0 \text{ as } t = 1/x \to -\infty$$

It follows that $f(x) = e^{1/x}$ has a vertical asymptote on the positive side of $x = 0$, but no vertical asymptote on the negative side. $\square$

iv. *(Horizontal asymptotes)* Let's check:

$$\lim_{x\to +\infty} e^{1/x} = 1 \quad \text{since } \frac{1}{x} \to 0 \text{ as } x \to +\infty \text{ and } e^t \to 1 \text{ as } t = 1/x \to 0$$
$$\lim_{x\to -\infty} e^{1/x} = 1 \quad \text{since } \frac{1}{x} \to 0 \text{ as } x \to -\infty \text{ and } e^t \to 1 \text{ as } t = 1/x \to 0$$

Thus $f(x) = e^{1/x}$ has a horizontal asymptote of $y = 1$ in both directions. $\Box$

v. *(Maxima and minima)* $f'(x) = \dfrac{d}{dx} e^{1/x} = e^{1/x} \cdot \dfrac{d}{dx}\left(\dfrac{1}{x}\right) = e^{1/x}\left(\dfrac{-1}{x^2}\right) = -\dfrac{e^{1/x}}{x^2}$ is, like $f(x) = e^{1/x}$, defined and continuous for all $x \neq 0$. Note that since $e^{1/x} > 0$ and $x^2 > 0$ for all $x \neq 0$, $f'(x) < 0$ for all $x \neq 0$. It follows that $f(x)$ is decreasing for all x for which it is defined; in particular, it has no critical points and no local maxima or minima.

vi. *(Curvature and inflection)* First,

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left(-\frac{e^{1/x}}{x^2}\right) = -\frac{\left(\frac{d}{dx}e^{1/x}\right)x^2 - e^{1/x}\left(\frac{d}{dx}x^2\right)}{\left(x^2\right)^2} = -\frac{-\frac{e^{1/x}}{x^2}x^2 - 2xe^{1/x}}{x^4} \\ &= -\frac{-e^{1/x} - 2xe^{1/x}}{x^4} = \frac{(1+2x)e^{1/x}}{x^4}, \end{aligned}$$

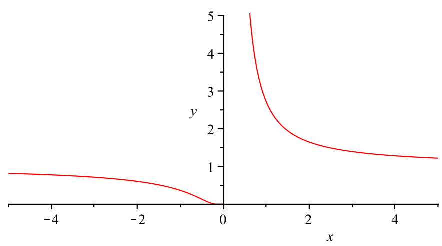
which is defined, just as $f(x)$ and $f'(x)$ are, for all $x \neq 0$. Note that since $e^{1/x} > 0$ whenever it is defined, $f''(x) = 0$ exactly when $1 + 2x = 0$, *i.e.* when $x = -\frac{1}{2}$. Since $x^4 > 0$ for all x, we also have that $f''(x) = \frac{(1+2x)e^{1/x}}{x^4} \lessgtr 0$ exactly when $1 + 2x \lessgtr 0$. Putting this information in the usual table gives us:

x	$\left(-\infty, \frac{1}{2}\right)$	$\frac{1}{2}$	$\left(\frac{1}{2}, 0\right)$	0	$(0, \infty)$
$f''(x)$	$-$	0	$+$	undef.	$+$
$f(x)$	$\frown$	infl. pt.	$\smile$	undef.	$\smile$

Thus $f(x)$ has one inflection point, at $x = \frac{1}{2}$. $\Box$

vii. *(Graph)* Here it is, at last, courtesy of `Maple`:

```
> plot(exp(1/x),x=-5..5,y=0..5)
```

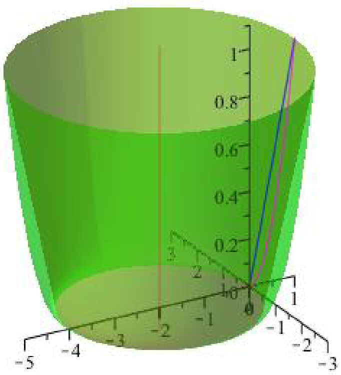


$\blacksquare$

7. Sketch the solid obtained by revolving the region between $y = x$ and $y = x^2$, for $0 \leq x \leq 1$, about the line $x = -2$ and find its volume.

SOLUTION. Here is `Maple`'s idea of a sketch of the solid:

```
> with(Student[Calculus1]):
> VolumeOfRevolution(x,x^2,x=0..1,axis=vertical,distancefromaxis=-2,
```



`output=plot,transparency=0.75,title=" ")`

Since we used the disk/washer method in the solution to **3g**, we'll use the cylindrical shell method here to compute the volume of the solid. Note that if $0 \leq x \leq 1$, then $x^2 \leq x$, so the height of the shell at x is given by $h = x - x^2$. The shell at x is obtained by revolving a vertical cross-section of the original region about the line $x = -2$, so the radius of the shell at x is given by $r = x - (-2) = x + 2$. Plugging these into the volume formula for the cylindrical shell method gives:

$$
\begin{aligned}
V &= \int_0^1 2\pi r h \, dx = \int_0^1 2\pi(x+2)\left(x - x^2\right)\, dx = 2\pi \int_0^1 \left(-x^3 - x^2 + 2x\right)\, dx \\
&= 2\pi\left(-\frac{x^4}{4} - \frac{x^3}{3} + x^2\right)\Bigg|_0^1 = 2\pi\left(-\frac{1^4}{4} - \frac{1^3}{3} + 1^2\right) - 2\pi\left(-\frac{0^4}{4} - \frac{0^3}{3} + 0^2\right) \\
&= 2\pi\frac{5}{12} - 2\pi 0 = \frac{5\pi}{6} \qquad \blacksquare
\end{aligned}
$$

Part K. Do *one* (1) of **8** or **9**. *[15 = 1 × 15 each]*

8. Let $f(x) = \dfrac{1}{(2+x)^2}$.

- **a.** Use Taylor's formula to find the Taylor series at 0 of $f(x)$. *[10]*
- **b.** Find the radius and interval of convergence of this Taylor series. *[5]*
- **c.** *[Bonus!]* Find the Taylor series at 0 of $f(x)$ without using Taylor's formula. *[1]*

Solutions. **a.** We'll grind out derivatives of $f(x) = \dfrac{1}{(2+x)^2} = (2+x)^{-2}$ at 0, looking for patterns we can plug into Taylor's formula.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(2+x)^{-2}$	$2^{-2} = \frac{1}{4}$
1	$-2(2+x)^{-3}$	$(-1)2 \cdot 2^{-3} = -\frac{1}{4}$
2	$(-2)(-3)(2+x)^{-4}$	$(-1)^2 2 \cdot 3 \cdot 2^{-4} = \frac{3}{8}$
3	$(-2)(-3)(-4)(2+x)^{-5}$	$(-1)^3 2 \cdot 3 \cdot 4 \cdot 2^{-5} = -\frac{3}{4}$
$\vdots$	$\vdots$	$\vdots$
n	$(-2)(-3)\cdots(-n-1)(2+x)^{-n-2}$	$(-1)^n (n+1)! 2^{-n-2} = \frac{(-1)^n (n+1)!}{2^{n+2}}$
$\vdots$	$\vdots$	$\vdots$

Plugging $f^{(n)}(0) = \dfrac{(-1)^n (n+1)!}{2^{n+2}}$ into Taylor's formula tells us that the Taylor series at 0 of $f(x) = \dfrac{1}{(2+x)^2}$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{\frac{(-1)^n (n+1)!}{2^{n+2}}}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{2^{n+2}} x^n . \qquad \blacksquare$$

b. To find the radius of convergence, we try the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}((n+1)+1)}{2^{(n+1)+2}} x^{n+1}}{\frac{(-1)^n (n+1)}{2^{n+2}} x^n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}\left((n+1)+1\right)}{2^{(n+1)+2}} x^{n+1} \cdot \frac{2^{n+2}}{(-1)^n (n+1) x^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{(-1)x(n+2)}{2(n+1)} \right| = \frac{|x|}{2} \lim_{n\to\infty} \frac{n+2}{n+1} \\
&= \frac{|x|}{2} \lim_{n\to\infty} \frac{n+2}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \frac{|x|}{2} \lim_{n\to\infty} \frac{1+\frac{2}{n}}{1+\frac{1}{n}} = \frac{|x|}{2} \cdot \frac{1+0}{1+0} = \frac{|x|}{2}
\end{aligned}$$

It follows that the Taylor series of $f(x)$ converges (absolutely) when $\frac{|x|}{2} < 1$, *i.e.* when $|x| < 2$, and diverges when $\frac{|x|}{2} > 1$, *i.e.* when $|x| > 2$. The radius of convergence of the series is therefore $R = 2$.

To find the interval of convergence we need to determine whether the series converges or diverges at $x = \pm R = \pm 2$. First, when $x = -2$, we get the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+2}} (-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^n 2^n}{2^{n+2}} = \sum_{n=0}^{\infty} \frac{1}{2^2} = \sum_{n=0}^{\infty} \frac{1}{4} ,$$

which diverges by the Divergence Test since $\displaystyle\lim_{n\to\infty} \frac{1}{4} = \frac{1}{4} \neq 0$. Second, when $x = +2$, we get the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+2}} 2^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{4} ,$$

which also diverges by the Divergence Test since $\lim_{n\to\infty} \frac{(-1)^n}{4}$ does not exists and hence is $\neq 0$. It follows that the interval of convergence of the Taylor series of $f(x)$ is $(-2,2)$. ■

c. There's a reason this one is a bonus – you have to put together several different things:

$$\begin{aligned}
\frac{1}{(2+x)^2} &= (2+x)^{-2} = -\frac{d}{dx}(2+x)^{-1} = \frac{d}{dx}\left(\frac{-1}{2+x}\right) = \frac{d}{dx}\left(\frac{-1}{2+x}\cdot\frac{\frac{1}{2}}{\frac{1}{2}}\right) \\
&= \frac{d}{dx}\left(\frac{-\frac{1}{2}}{1+\frac{x}{2}}\right) = \frac{d}{dx}\left(\frac{-\frac{1}{2}}{1-\left(-\frac{x}{2}\right)}\right) = \frac{d}{dx}\left(-\frac{1}{2}+\frac{x}{4}-\frac{x^2}{8}+\frac{x^3}{16}-\cdots\right) \\
&= \frac{d}{dx}\left(-\frac{1}{2}\right) + \frac{d}{dx}\left(\frac{x}{4}\right) - \frac{d}{dx}\left(\frac{x^2}{8}\right) + \frac{d}{dx}\left(\frac{x^3}{16}\right) - \cdots \\
&= 0 + \frac{1}{4} - \frac{2x}{8} + \frac{3x^2}{16} - \cdots + \frac{(-1)^n(n+1)x^n}{2^{n+2}} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)x^n}{2^{n+2}}
\end{aligned}$$

If a function is equal to a power series, then that series is its Taylor series, so ... ■

9. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (z-2)^n}{2^n}$, where z is an unknown.

a. Determine for which values of z the series converges. *[10]*

b. Find a function $g(z)$ equal to this series when it converge. *[5]*

SOLUTIONS. **a.** Just for fun, we'll use the Root Test. (The Ratio Test works well here, too.)

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n(z-2)^n}{2^n}\right|} = \lim_{n\to\infty} \sqrt[n]{\frac{|z-2|^n}{2^n}} = \lim_{n\to\infty} \frac{|z-2|}{2} = \frac{|z-2|}{2}$$

It follows that the series converges when $\frac{|z-2|}{2} < 1$, *i.e.* $|z-2| < 2$ (that is, $0 < z < 4$), and diverges when $\frac{|z-2|}{2} > 1$, *i.e.* $|z-2| > 2$ (that is, $z < 0$ or $z > 4$).

This still leaves the question of what happens when $|z-2| = 2$, that is, when $z = 0$ or $z = 4$. When $z = 0$, we get the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n(0-2)^n}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n 2^n}{2^n} = \sum_{n=1}^{\infty} 1\,,$$

which diverges by the Divergence Test because $\lim_{n\to\infty} 1 = 1 \neq 0$. When $z = 4$, we get the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n(4-2)^n}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{2^n} = \sum_{n=1}^{\infty} (-1)^n\,,$$

which diverges by the Divergence Test because $\lim_{n\to\infty} (-1)^n$ does not exist, so does $\neq 0$.

Thus the series converges exactly when $0 < z < 4$, and diverges otherwise. ∎

NOTE: **a** is much easier if you notice that the series is a geometric series with $a = r = -(z-2)/2$.

b. As noted just above, the given series is a geometric series with $a = r = -(z-2)/2$. When it converges, its sum is therefore

$$g(z) = \frac{a}{1-r} = \frac{-\frac{z-2}{2}}{1-\left(-\frac{z-2}{2}\right)} = \frac{-(z-2)}{2\left(1+\frac{z-2}{2}\right)} = \frac{2-z}{2+z-1} = \frac{2-z}{1+z}\,. \qquad ∎$$

[Total = 100]

Part Z. Bonus problems! Do them (or not – less for me to mark! :-), if you feel like it.

0. Recall that an integer greater than 1 is a prime number if it has no positive integer factors other than itself and 1. Does the polynomial $p(x) = x^2 + x + 41$ always give you a prime number as its output whenever x is an integer greater than or equal to zero? Explain why or why not. *[1]*

SOLUTION. It does not always give you a prime number when x is an integer greater than or equal to zero, though it does give you prime numbers for $x = 0, 1, \dots, 40$. However, $p(41) = 41^2 + 41 + 41 = 41(41 + 1 + 1) = 41 \cdot 43$, so it is not prime. ■

00. Write a haiku touching on calculus or mathematics in general. *[2]*

haiku?

seventeen in three:
five and seven and five of
syllables in lines

I HOPE YOU HAVE EVEN MORE FUN THIS SUMMER
THAN YOU DID IN THIS COURSE!