

Mathematics 1101Y – Calculus I: Functions and calculus of one variable

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Solutions to Assignment #2 Off on a poetical tanchent ...

Mathematicians in medieval India traditionally wrote up much of their work in verse. For example, here is a problem posed by Bhaskara (*c.* 1114–1185 A.D.) in a book dedicated to his daughter Lilavati:

The square root of half the number of bees in a swarm
Has flown out upon a jasmine bush;
Eight ninths of the swarm has remained behind;
And a female bee flies about a male who is buzzing inside a lotus flower;
In the night, allured by the flower's sweet odour, he went inside it
And now he is trapped!
Tell me, most enchanting lady, the number of bees.

1. Restate the problem given above as an equation. [1]

SOLUTION. If x is the total number of bees in the swarm, the problem tells us that $\sqrt{\frac{x}{2}}$ of them have flown to the jasmine bush, $\frac{8}{9}x$ have remained behind, and 2 more are in or around the lotus flower. Thus

$$x = \sqrt{\frac{x}{2}} + \frac{8}{9}x + 2$$

is the equation given in the problem. ■

2. Solve the equation you obtained in 1 by hand. [1]

SOLUTION. Here goes! We first rearrange the equation to help isolate and get rid of the square root, and then eliminate all the fractions among the coefficients:

$$\begin{aligned} x = \sqrt{\frac{x}{2}} + \frac{8}{9}x + 2 &\implies \sqrt{\frac{x}{2}} = \frac{1}{9}x - 2 \implies \frac{x}{2} = \left(\frac{1}{9}x - 2\right)^2 = \frac{1}{81}x^2 - \frac{4}{9}x + 4 \\ &\implies \frac{1}{81}x^2 - \frac{17}{18}x + 4 = 0 \implies 2x^2 - 153x + 648 = 0 \end{aligned}$$

... and then solve the resulting quadratic equation using the quadratic formula:

$$\begin{aligned} x &= \frac{-(-153) \pm \sqrt{(-153)^2 - 4 \cdot 2 \cdot 648}}{2 \cdot 2} = \frac{153 \pm \sqrt{23409 - 5184}}{4} \\ &= \frac{153 \pm \sqrt{18225}}{4} = \frac{153 \pm 135}{4} = \frac{288}{4} = \frac{72}{1} \\ &\quad \text{or } \frac{18}{4} = \frac{9}{2} \end{aligned}$$

Thus x , the number of bees in the swarm, is either 72 or $\frac{9}{2}$. Bhaskara didn't allow for fractional bees ... [But see the bonus problem!] ■

3. Solve the equation you obtained in 1 using Maple. [1]

Note: The basic tool you will need to do 3 is Maple's `solve` command, which has many options and variations. Make sure to ask for help if you need it!

SOLUTION. Plugging the original equation into Maple gives:

```
> solve(x=sqrt(x/2)+8*x/9+2,x);
```

72

That is, $x = 72$ is the solution to the equation described by Bhaskara.

On the other hand, plugging the equivalent quadratic equation into Maple gives:

```
> solve(2*x^2-153*x+648=0,x);
```

$\frac{9}{2}, 72$

That is, $x = \frac{9}{2}$ and $x = 72$ are the two solutions of the quadratic equation we obtained and then solved in **2**. Why doesn't Maple give $\frac{9}{2}$ as a solution of the original equation? Beats me! ■

Recall that the basic *hyperbolic trigonometric functions* include:

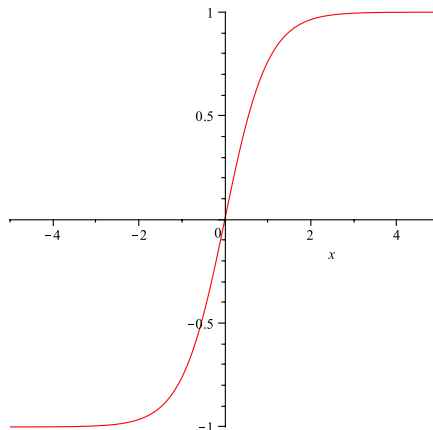
$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \cosh(x) = \frac{e^x + e^{-x}}{2} \quad \tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

There are connections between the hyperbolic and the regular trigonometric functions, some of which will become apparent when we study series. Your main task in this assignment will be to investigate $\tanh(x)$ and find its inverse.

4. Use Maple to graph $\tanh(x)$. [1]

SOLUTION. This ought to be pretty easy after Assignment #1:

```
> plot(tanh(x), x = -5 .. 5);
```



■

5. From your graph from **4**, what restrictions does the output of $\tanh(x)$ satisfy? Using the definition of $\tanh(x)$ given above, explain why its output has to satisfy these restrictions. [1]

SOLUTION. It is pretty obvious from the graph that $\tanh(x)$ has $y = 1$ as a horizontal asymptote in the +ve direction and $y = -1$ as a horizontal asymptote in the -ve direction.

To see why, note first that e^x and e^{-x} are both positive for any x . It follows that their difference must be smaller in magnitude than their sum, *i.e.* $|e^x - e^{-x}| < |e^x + e^{-x}|$, so, for any x ,

$$|\tanh(x)| = \left| \frac{e^x - e^{-x}}{e^x + e^{-x}} \right| = \frac{|e^x - e^{-x}|}{|e^x + e^{-x}|} < 1.$$

In particular, since

$$\lim_{x \rightarrow -\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x \rightarrow -\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x \rightarrow -\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = \lim_{x \rightarrow -\infty} \frac{\frac{d}{dx}(1 - e^{-2x})}{\frac{d}{dx}(1 + e^{-2x})}$$

(We can use l'Hôpital's Rule since $1 \pm e^{-2x} \rightarrow \pm\infty$ as $x \rightarrow -\infty$.)

$$= \lim_{x \rightarrow -\infty} \frac{2e^{-x}}{-2e^{-x}} = \lim_{x \rightarrow -\infty} \frac{2}{-2} = -1 \quad \text{and}$$

$$\lim_{x \rightarrow +\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x \rightarrow +\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^x}{e^x} = \lim_{x \rightarrow +\infty} \frac{e^{2x} - 1}{e^{2x} + 1} = \lim_{x \rightarrow +\infty} \frac{\frac{d}{dx}(e^{2x} - 1)}{\frac{d}{dx}(e^{2x} + 1)}$$

(We can use l'Hôpital's Rule since $e^{2x} - 1 \pm 1 \rightarrow +\infty$ as $x \rightarrow +\infty$.)

$$= \lim_{x \rightarrow +\infty} \frac{2e^x}{2e^x} = \lim_{x \rightarrow +\infty} \frac{2}{2} = 1,$$

it follows that $\tanh(x)$ has $y = -1$ as a horizontal asymptote to the left and $y = 1$ as a horizontal asymptote to the right. ■

6. Use Maple's ability to solve equations symbolically to find an expression for $\operatorname{arctanh}(x)$, the inverse function of $\tanh(x)$.** [3]

SOLUTION. Just plugging $y = \tanh(x)$ into the `solve` command and asking Maple to solve for x gives

```
> solve(y=tanh(x), x);
arctanh(x)
```

which isn't very useful to us. We therefore resort to the definition of $\tanh(x)$ and plug

$y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ into the `solve` command and ask it to solve for x :

```
> solve(y=(exp(x)-exp(-x))/(exp(x)+exp(-x)), x);
```

$$\ln \left(\frac{\sqrt{-(y-1)(1+y)}}{y-1} \right), \ln \left(-\frac{\sqrt{-(y-1)(1+y)}}{y-1} \right)$$

Note the use of `exp(x)` for e^x in the "classic worksheet" style command. Interchanging the roles of x and y , Maple's output tells us that $\ln \left(\frac{\sqrt{-(x-1)(1+x)}}{x-1} \right)$ and $\ln \left(-\frac{\sqrt{-(x-1)(1+x)}}{x-1} \right)$ are possible expressions for $\operatorname{arctan}(x)$. ■

** Our textbook uses the notation $\tanh^{-1}(x)$ for the inverse of $\tanh(x)$.

7. Work out an expression for $\operatorname{arctanh}(x)$ yourself. (If this is different from what **Maple** gave you in **6**, you may well be correct, but try to explain, if you can, why they amount to the same thing.) [2]

SOLUTION. Let $x = \tanh(y) = \frac{\sinh(y)}{\cosh(y)} = \frac{e^y - e^{-y}}{e^y + e^{-y}}$. We shall attempt to solve for y in terms of x :

$$\begin{aligned} x = \frac{e^y - e^{-y}}{e^y + e^{-y}} &\iff x(e^y + e^{-y}) = e^y - e^{-y} \iff x(e^y + e^{-y})e^y = (e^y - e^{-y})e^y \\ &\iff x((e^y)^2 + 1) = (e^y)^2 - 1 \iff x(e^y)^2 + x = (e^y)^2 - 1 \\ &\iff (x - 1)(e^y)^2 = -1 - x = -(x + 1) \\ &\iff (e^y)^2 = -\frac{x + 1}{x - 1} = \frac{1 + x}{1 - x} \iff e^y = \pm \sqrt{\frac{1 + x}{1 - x}} = \pm \left(\frac{1 + x}{1 - x}\right)^{1/2} \end{aligned}$$

Note that $e^y > 0$ for all y and $\sqrt{a} \geq 0$ for all a for which it makes sense (*i.e.* $a \geq 0$), so:

$$\iff e^y = + \left(\frac{1 + x}{1 - x}\right)^{1/2} \iff y = \ln \left(\left(\frac{1 + x}{1 - x}\right)^{1/2} \right) = \frac{1}{2} \ln \left(\frac{1 + x}{1 - x} \right)$$

That is, $\operatorname{arctanh}(x) = \frac{1}{2} \ln \left(\frac{1 + x}{1 - x} \right)$. Note that it makes sense only when $\frac{1 + x}{1 - x} > 0$, that is, when $1 - x > 0$ and $1 + x > 0$ or when $1 - x < 0$ and $1 + x < 0$. $1 - x > 0$ and $1 + x > 0$ boil down to $-1 < x < 1$, which matches the possible output of $\tanh(x)$, and $1 - x < 0$ and $1 + x < 0$ boil down to $1 < x$ and $x < -1$, which it is impossible to satisfy simultaneously.

How does this compare to the expressions given by **Maple** in the solution to **6**? We will examine when those make sense:

First, for $\ln \left(\frac{\sqrt{-(x-1)(1+x)}}{x-1} \right)$ to make sense, we need to have that $-(x-1)(1+x) > 0$ and $x-1 > 0$. Combining these two, we see that we must also have $-(1+x) = -1-x > 0$, *i.e.* $-1 > x$. This is impossible to satisfy simultaneously with $x-1 > 0$, *i.e.* $x > 1$, so this expression makes no sense for any real number x .

Second, for $\ln \left(-\frac{\sqrt{-(x-1)(1+x)}}{x-1} \right)$ to make sense, we need to have that $-(x-1)(1+x) > 0$ and $-(x-1) = -x+1 > 0$. Combining these two, we see that we must also have $1+x > 0$, *i.e.* $x > -1$. Since the condition $-x+1 > 0$ amounts to $1 > x$, the overall expression makes sense if $-1 < x < 1$. For such x we then have

$$\begin{aligned} \operatorname{arctanh}(x) &= \ln \left(-\frac{\sqrt{-(x-1)(1+x)}}{x-1} \right) = \ln \left(\frac{\sqrt{(1-x)(1+x)}}{1-x} \right) = \ln \left(\frac{\sqrt{1+x}}{\sqrt{1-x}} \right) \\ &= \ln \left(\sqrt{\frac{1+x}{1-x}} \right) = \ln \left(\left(\frac{1+x}{1-x} \right)^{1/2} \right) = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right), \end{aligned}$$

which is just what we obtained above.

You might ask why **Maple** gave an expression that wasn't a solution, at least for any real values of x . The reason is probably that some of the algorithms **Maple** uses to solve equations exploit the properties of complex numbers, which have various advantages for algebra. (Notably, every polynomial can be factored into linear factors if one is allowed to use complex numbers; complex numbers also make for very close connections between trigonometric and exponential functions, as we will see later in the course.) Some ways of trying to do **6** may in fact get **Maple** to give you expressions that explicitly involve complex numbers. ■

Bonus. What does Bhaskara's problem have to do with a *Monty Python* sketch? [0.5]

SOLUTION. The 1972 album *Monty Python's Previous Record* from the British comedy group *Monty Python* includes the song *Eric the Half-a-bee*, where it concludes a variant of their classic *Fish License* sketch. You can find the song on YouTube at:

<http://www.youtube.com/watch?v=MlrsqGal64w>

This song includes the immortal line, "*For half a bee, philosophically, must ipso facto half not be.*" Some of the group had, it seems, studied philosophy ... ■