Mathematics 1101Y – Calculus I: functions and calculus of one variable
TRENT UNIVERSITY, 2010–2011

Assignment #7
Bad function!
Due on Friday, 21 January, 2011.

The following function was mentioned in class last term:

$$f(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$$

It is a standard example in calculus-related courses of a function which is not continuous at any point:

1. Using the $\varepsilon - \delta$ definition of limits, explain why $\lim_{x \to a} f(x)$ is undefined for any a. *[5]*

Hint: No matter how close together $c < d$ are, (c, d) will contain both rational numbers and irrational numbers.

It follows from **1** that $f(x)$ can't be continuous at any point: no limits, no continuity. Note that because the output of $f(x)$ is bounded, it has no asymptotes: all of its discontinuities are jump discontinuities.

$f(x)$ is also badly behaved from the standpoint of Riemann integration:

2. Using the definition of the definite integral given in §5.2, explain why $\int_a^b f(x)\,dx$ is undefined for any $a < b$. *[5]*

Hint: No matter how close together $c < d$ are, (c, d) will contain both rational numbers and irrational numbers. *[Yes, it's the same hint!]*

Note that Theorem 3 of §5.2 guarantees that any function with only finitely many jump discontinuities (and no asymptotes) in $a, b]$ will be integrable on $[a, b]$.

More sophisticated definitions of integration (generalized Riemann integration and Lebesgue integration, for two) can make sense of $\int_a^b f(x)\,dx$.

Bonus. What *should* $\int_a^b f(x)\,dx$ be equal to? Why (very roughly)? *[1]*