Math 1100 — Calculus, Quiz #7B — 2009-11-27

1. Let $f(x) := \sqrt[3]{1+x}$ for all $x \geq 0$.

(10) (a) Show that $f'(x) < \frac{1}{3}$ for all $x > 0$.

Solution: For all $x > 0$, we have

$$f'(x) \quad = \quad \frac{1}{3(1+x)^{2/3}} \quad < \quad \frac{1}{3},$$

where the inequality is because $(1+x)^{2/3} > 1$ because $(1+x) > 1$ because $x > 0$. □

(40) (b) Show that $f(x) \leq 1 + \frac{1}{3}x$ for all $x > 0$.

Solution: (by contradiction) Let $f(x) := \sqrt[3]{1+x}$.

Suppose $f(x) > 1 + \frac{1}{3}x$ for some $x > 0$. Then the Mean Value Theorem says there is some $y \in (0, x)$ such that

$$f'(y) \quad = \quad \frac{f(x) - f(0)}{x - 0} \quad = \quad \frac{f(x) - 1}{x} \quad > \quad \frac{1 + \frac{1}{3}x - 1}{x} \quad = \quad \frac{\frac{1}{3}x}{x} \quad = \quad \frac{1}{3}.$$

But this contradicts the conclusion of part (a). By contradiction, we cannot have $f(x) > 1+\frac{1}{3}x$; hence $f(x) \leq 1 + \frac{1}{3}x$ □

(50) 2. Define $f : [0,3] \longrightarrow \mathbb{R}$ by $f(x) := x^3 - 3x + 1$. Find the global maximum and global minimum of f on the interval $[0,3]$.

Solution: If $f(x) := x^3 - 3x + 1$, then $f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$, for all $x \in (0,3)$, and f is differentiable everywhere on $(0,3)$. Thus the critical points of f are at the roots of f' —namely at ± 1. However, -1 is not in the domain, so we ignore it. To identify the global maximum and global minimum of f we compute the value of f at the critical point 1, and also at both endpoints. We have:

$$\begin{aligned} f(1) &= 1 - 3 + 1 &= -1; \\ f(3) &= 27 - 9 + 1 &= 19; \\ f(0) &= 0 + 0 + 1 &= 1. \end{aligned}$$

Thus, the $\boxed{\text{global maximum is at the endpoint } 3,}$ and the $\boxed{\text{global minimum is at the critical point } 1.}$ □