Math 1100 — Calculus, Quiz #17B — 2010-04-01

1. Consider the parametric curve parameterized by the functions $x(t) := e^t + e^{-t}$ and $y(t) := 5 - 2t$, for all $t \in [-1, 1]$.

(10) (a) Compute $x'(t)$ and $y'(t)$.

Solution: $x'(t) = e^t - e^{-t}$ and $y'(t) = -2$. □

(10) (b) Find an expression for the *slope* of the curve at the point $(x(t), y(t))$, as a function of t.

Solution: $\text{slope}(t) \;=\; \dfrac{y'(t)}{x'(t)} = \dfrac{-2}{e^t - e^{-t}} = \dfrac{2}{e^{-t} - e^t}$. □

(10) (c) For what value(s) of t is the tangent line of the curve *horizontal*? For what value(s) of t is the tangent line of the curve *vertical*?

Solution: The tangent is horizontal when $\text{slope}(t)$ is zero, which never occurs for any value of t.

The tangent is vertical when $\text{slope}(t)$ is infinite, which occurs when $t = 0$. □

(20) (d) Compute the *arc length* of the curve between $t = 0$ and $t = 1$.

Solution:

$$\begin{aligned}
x'(t)^2 + y'(t)^2 &= \left(e^t - e^{-t}\right)^2 + (-2)^2 \;=\; e^{2t} - 2 + e^{-2t} \;+\; 4 \\
&= e^{2t} + 2 + e^{-2t} \;=\; (e^t + e^{-t})^2. \\
\text{Thus, } \sqrt{x'(t)^2 + y'(t)^2} &= e^t + e^{-t}. \\
\text{Thus, arclength} &= \int_0^1 \sqrt{x'(t)^2 + y'(t)^2}\, dt \;=\; \int_0^1 e^t + e^{-t}\, dt \;=\; e^t - e^{-t}\Big|_{t=0}^{t=1} \\
&= \left(e - \frac{1}{e}\right) - (1 - 1) \;=\; \;=\; \boxed{e - \frac{1}{e}.}
\end{aligned}$$

□

2. Let $a_n := \dfrac{n+1}{3n^2}$ for all $n \in \mathbb{N}$. (So $a_1 = \frac{2}{3}$, $a_2 = \frac{1}{4}$, $a_3 = \frac{4}{27}$, etc.)

(10) (a) Is the sequence $\{a_n\}_{n=1}^{\infty}$ convergent? If so, what is its limit?

Solution: Yes, the sequence is convergent. We have:

$$\lim_{n\to\infty} \frac{n+1}{3n^2} \;=\; \lim_{n\to\infty} \left(\frac{1}{3n} + \frac{1}{3n^2}\right) \;=\; \left(\frac{1}{3} \lim_{n\to\infty} \frac{1}{n}\right) + \left(\frac{1}{3} \lim_{n\to\infty} \frac{1}{n^2}\right) \;=\; 0+0 \;=\; \boxed{0.}$$

□

(15) (b) Is the series $\sum_{n=1}^{\infty} a_n$ convergent or divergent? If it is convergent, what is its sum?

Solution: The series is $\boxed{\text{divergent.}}$ To see this, observe that

$$\sum_{n=1}^{\infty} \frac{n+1}{3n^2} \quad = \quad \sum_{n=1}^{\infty} \left(\frac{1}{3n} + \frac{1}{3n^2} \right) \quad = \quad \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n} \quad + \quad \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Now, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. However, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges* (it is the 'Harmonic series'). Both summands have the same sign (positive), so the sum of the two series also diverges. □

3. Let $x \in \mathbb{R}$ be unknown, and consider the series $\sum_{n=0}^{\infty} \left(\frac{x}{3} \right)^n$.

(15) (a) For what values of x (if any) does this series converge?

Solution: The Geometric Series Theorem says that $\sum_{n=0}^{\infty} \left(\frac{x}{3} \right)^n$ converges if and only if $\left| \frac{x}{3} \right| < 1$. This occurs if and only if $|x| < 3$. Thus, the series converges for all $x \in (-3, 3)$. □

(10) (b) Suppose we define the function $f(x) := \sum_{n=0}^{\infty} \left(\frac{x}{3} \right)^n$, for all x where this series converges. Find a nice simple expression for $f(x)$ (not involving an infinite sum).

Solution: For those x where the series *does* converge, the Geometric Series Theorem says that

$$\sum_{n=0}^{\infty} \left(\frac{x}{3} \right)^n \quad = \quad \frac{1}{1 - \frac{x}{3}} \quad = \quad \frac{1}{\frac{3-x}{3}} \quad = \quad \boxed{\frac{3}{3-x}.}$$

□