Math 1100 — Calculus, HW #2 — Due Monday, 2010-01-11

The economics of ice cream

Solutions

'Common mistakes' are indicated in your marked assignmend with circled numbers, e.g. ①, ②, ③, etc. These labels are explained in the remarks following the solutions to each question.

1. Alice is contemplating how much ice cream to consume. Her enjoyment of ice cream is described by a *utility function* $U : \mathbb{R}_+ \longrightarrow \mathbb{R}$. If she consumes q units of ice cream, then her 'utility' (i.e. her happiness, satisfaction, enjoyment, etc.) will be $U(q)$. Assume U is differentiable everywhere on $\mathbb{R}_+$. The derivative $U'(q)$ is Alice's *marginal utility* for ice cream —intuitively, $U'(q)$ measures the additional utility she would obtain by eating a single spoonful beyond the amount q she is currently consuming. Assume U' is a *decreasing* function.[1]

   The money which Alice does *not* spend on ice cream can be spent on other goods and services, which also produce enjoyment. This is described by another utility function $V : \mathbb{R}_+ \longrightarrow \mathbb{R}$. If Alice spends r dollars of money on other goods beside ice cream, she will obtain $V(r)$ units of utility. For simplicity, we assume that $V(r) = r$ for all $r \in \mathbb{R}_+$. [2]

   Suppose the price of ice cream is p dollars per unit (thus, q units of ice cream cost pq dollars). Thus, if Alice begins with r dollars, and she buys q units of ice cream, and spends the remaining $(r - pq)$ dollars for other goods, then her total utility will be $W(q) := U(q) + V(r - pq) = U(q) + r - pq$. We assume that Alice purchases the
$\left(\frac{25}{200}\right)$ quantity q which *maximizes* her utility $W(q)$.

   Show that q maximizes W if and only if $U'(q) = p$.

   (Assume that $U'(r) < p < U'(0)$.)

Solution: Observe that $W'(q) = U'(q) - p$, and W is differentiable everywhere on $\mathbb{R}_+$ because U is differentiable everywhere on $\mathbb{R}_+$. Fermat's theorem says that any extremum of W occurs at a critical point —that is, at a place where $W'(q) = 0$. But $W'(q) = U'(q) - p$, so q is a critical point if and only if $U'(q) = p$.

Is this critical point a maximum or a minimum? Recall that U' is a decreasing function. Thus, W' is also decreasing. Thus, $W'(r) > 0$ for all $r < q$, and $W'(r) < 0$ for all $r > q$. Thus, q is a global maximum of W.

① A few people argued that $W''(q) = U''(q)$ and $U''(q) < 0$ because U' is assumed to be decreasing. This argument works fine as long as U' is differentiable —however, in general there is no reason

[1] This is the assumption of *declining marginal utility*. It means Alice becomes satiated –the 100th spoonful of ice cream produces less pleasure than the first spoonful did.

[2] That is, we measure utility in 'dollar units'. This is just a convenient simplification —it does not represent some philosophical commitment to the idea that money=happiness.

to assume that U' is differentiable, nor is this assumption necessary to solve the problem. This mistake cost 5 points

② Many people showed that q is a critical point of W if $U'(q) = p$, but they forgot to show that this critical point was a *maximum* (e.g. by arguing that W' is decreasing). In general, a critical point could be a maximum, a minimum, or even a flat inflection point. This mistake cost 10 points.

Remark. In my email, I mentioned that you must also assume that $U'(0) > p$, and asked, "What would happen if $U'(0) \leq p$?" Here's what would happen: Alice's utility-maximizing choice would be to buy *zero* units of ice-cream (since she can't buy a 'negative' amount).

Also, note that we must assume that $U'(r) < p$. (Only one person spotted this).If $U'(r) > p$, then Alice's utility-maximizing choice is to spend *all* of her money (i.e. all r dollars) on icecream. Indeed, if Alice was allowed to go into debt, then she would spend *more* than r dollars on icecream (e.g. by deploying her credit card, mortgaging her house, etc.) □

2. $(\frac{25}{200})$ Let $Q_d(p)$ be the utility-maximizing quantity of ice cream you found in question #1. Note that $Q_d(p)$ depends on the price p. Show that Q_d is *decreasing* as a function of p. [3]

Solution: For any $q \in \mathbb{R}_+$ and $p \in \mathbb{R}_+$, question #1 shows that $Q_d(p) = q$ if and only if $U'(q) = p$. In other words, Q_d is the *inverse function* of U'. But U' is a decreasing function. Thus, Q_d is also decreasing, because the inverse function of any decreasing function is decreasing.

To see this, suppose $p_1 < p_2$. Let $q_1 = Q_d(p_1)$ and $q_2 = Q_d(p_2)$. We must show that $q_1 > q_2$. But we have $U'(q_1) = p_1 < p_2 = U'(q_2)$. But U' is decreasing, so this means that $q_1 > q_2$, as claimed.

③ Many people simply copied the hint in my email and wrote, "Q_d is the inverse of U', and the inverse of a decreasing function is decreasing," without providing any justification for either of these assertions. Neither one of these observations is very deep, but nevertheless I still wanted to see some mathematical justification for each of them. □

3. Bob runs an ice cream factory. Let $C(q)$ be the total cost of producing q units of ice cream. Assume C is differentiable everywhere on $\mathbb{R}_+$. The derivative $C'(q)$ is called the ***marginal cost*** (it is the additional cost of producing one more spoonful of ice cream if you are already producing q units). We assume that C' is an *increasing* function.[4]

Suppose Bob can sell all the ice cream he produces at the market price p. Thus, if he produces (and sells) q units, his total revenue is pq dollars, while his total cost of production is $C(q)$ dollars, so his ***profit*** is $\Pi(q) := pq - C(q)$. We assume Bob chooses
$(\frac{25}{200})$ q so as to *maximize* his profit $\Pi(q)$.

Show that q maximizes Π if and only if $C'(q) = p$.

[3] Thus, the more expensive ice cream becomes, the less Alice will want to buy.

[4] This is the assumption of *increasing marginal costs*, or *decreasing return to scale*. Intuitively, the more ice cream you are already producing, the harder it is to squeeze out an additional unit of productivity, due to bottlenecks in the production line, machinery overheating, depletion of local suppliers, labour shortages, etc.

Solution: Observe that $\Pi'(q) = p - C'(q)$, and Π is differentiable everywhere on $\mathbb{R}_+$ because C is differentiable everywhere on $\mathbb{R}_+$. Fermat's theorem says that any extremum of Π occurs at a critical point —that is, at a place where $\Pi'(q) = 0$. But $\Pi'(q) = p - C'(q)$, so q is a critical point if and only if $C'(q) = p$.

Is this critical point a maximum or a minimum? Recall that C' is a increasing function. Thus, $-C'$ is *decreasing*. Thus, Π' is decreasing. Thus, $\Pi'(r) > 0$ for all $r < q$, and $\Pi'(r) < 0$ for all $r > q$. Thus, q is a global maximum of Π.

④ Many people showed that q is a critical point of Π if $C'(q) = p$, but they forgot to show that this critical point was a *maximum* (e.g. by arguing that Π' is decreasing). In general, a critical point could be a maximum, a minimum, or even a flat inflection point. This mistake cost 10 points. □

$(\frac{25}{200})$ 4. Let $Q_s(p)$ be the profit-maximizing production of ice cream you found in question #3. Note that $Q_s(p)$ depends on the price p. Show that Q_s is *increasing* as a function of p. [5]

Solution: For any $q \in \mathbb{R}_+$ and $p \in \mathbb{R}_+$, question #3 shows that $Q_s(p) = q$ if and only if $C'(q) = p$. In other words, Q_d is the *inverse function* of C'. Now, C' is increasing. Thus, Q_s is also increasing, because the inverse function of any increasing function is increasing.

To see this, suppose $p_1 < p_2$. Let $q_1 = Q_s(p_1)$ and $q_2 = Q_s(p_2)$. We must show that $q_1 < q_2$. But we have $C'(q_1) = p_1 < p_2 = C'(q_2)$. But C' is increasing, so this means that $q_1 < q_2$, as claimed. □

5. The function Q_d in question #2 is Alice's *demand function* for ice cream. The *market demand function* $\overline{Q}_d$ is obtained by adding together the demand functions of all ice cream consumers. It is a decreasing function (because of question #2). The function Q_s in question #4 is Bob's *supply function*. The *market supply function* $\overline{Q}_s$ is obtained by adding together the supply functions of all ice cream producers. It is an increasing function (because of question #4).

$(\frac{25}{200})$ Suppose that $\overline{Q}_d$ and $\overline{Q}_s$ are both continuous. Suppose that $\overline{Q}_d(0) > \overline{Q}_s(0)$. [6] Also, suppose that there is some large price $\overline{p}$ such that $\overline{Q}_d(\overline{p}) < \overline{Q}_s(\overline{p})$. [7]

Show that there exists a unique *equilibrium price*[8] $p^* \in [0, \overline{p}]$ such that $\overline{Q}_d(p^*) = \overline{Q}_s(p^*)$.

(*Hint:* Consider the *excess demand* function $E(p) := \overline{Q}_d(p) - \overline{Q}_s(p)$. Show that $E(p)$ has exactly one zero.)

Solution: *Existence.* Let $E(p) := \overline{Q}_d(p) - \overline{Q}_s(p)$ for all $p \in \mathbb{R}_+$. Then E is a continuous function, because $\overline{Q}_d$ and $\overline{Q}_s$ are continuous. Now, $E(0) > 0$ because $\overline{Q}_d(0) > Q(s)$. Meanwhile $E(\overline{p}) < 0$ because $\overline{Q}_d(\overline{p}) < \overline{Q}_s(\overline{p})$. Thus, the Intermediate Value Theorem says there exists at least one $p^* \in (0, \overline{p})$ such that $E(p^*) = 0$, which means $\overline{Q}_d(p^*) = \overline{Q}_s(p^*)$.

[5] Thus, the higher the market price for ice cream is, the more Bob will want to produce.
[6] Intuitively: if ice cream is very cheap, then demand will exceed supply —there will be a *shortage*.
[7] Intuitively: if ice cream is very expensive, then supply will exceed demand: there will be a *glut*.
[8] A basic tenet of microeconomics is that the market will usually converge to this equilibrium price.

Uniqueness. $E(p)$ is a decreasing function (because $\overline{Q}_d$ is decreasing and $\overline{Q}_s$ is increasing). Thus, if $p < p^*$, then $E(p) > 0$, whereas if $p > p^*$, then $E(p) < p^*$. Thus, p^* is the only zero of E, so p^* is the *unique* price such that $\overline{Q}_d(p^*) = \overline{Q}_s(p^*)$.

Remark. Some people proved 'existence' but not 'uniqueness'. Other people proved 'uniqueness' but not 'existence'. These people lost marks. □

6. The model in questions #3-#5 is a ***competitive market***, where Bob is only one of many *competing* ice cream makers, and he cannot influence the market price p —he can only respond to p by picking a profit-maximizing production level. But not all markets are competitive; some are ***monopolies*** with only a single producer.[9]

   Let P_d be the inverse function of $\overline{Q}_d$ (that is: if $\overline{Q}_d(p) = q$, then $P_d(q) = p$). If monopolist Bob chooses to produce q units of ice cream, then he can sell all these units if he sets the price to $P_d(q)$. His total revenue will then be $q \cdot P_d(q)$. Meanwhile, the cost of producing q units is still $C(q)$. Thus, Bob's profit will be $\Pi(q) := q \cdot P_d(q) - C(q)$.
$(\frac{25}{200})$ Again, we assume Bob chooses q so as to maximize Π. Assume P_d is differentiable.

   Show: if q maximizes Π, then $P_d(q) = C'(q) - q \cdot P_d'(q)$.

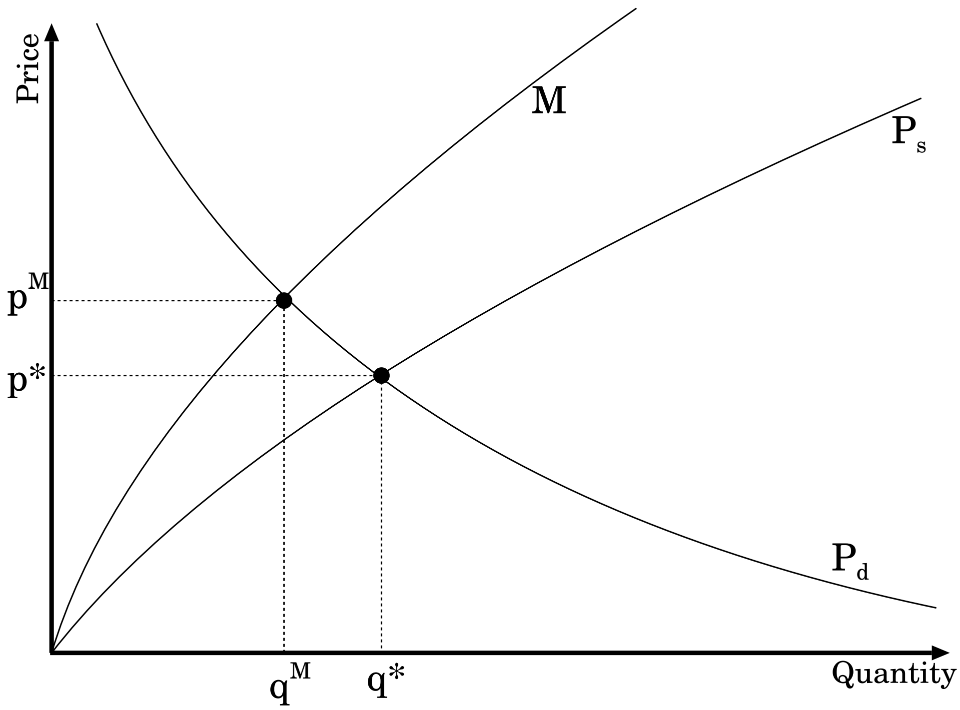
Solution: If $\Pi(q) := q \cdot P_d(q) - C(q)$, then $\Pi'(q) = q \cdot P_d'(q) + P_d(q) - C'(q)$. As in parts #1 and #3, the maximizer occurs at the critical point, which is the point q such that $\Pi'(q) = 0$ —that is, where $P_d(q) = C'(q) - q \cdot P_d'(q)$.

Remark. I had originally formulated these question as an 'if and only if' statement. However, when marking the question, I realized that I hadn't given you enough information to show that the critical point of Π is necessarily a maximizer. The problem is the same as 'Common Mistakes' ② and ④: you must show that Π' is decreasing. To do this, you need some sort of additional hypothesis on P_d' —e.g. you could solve the problem if you assumed that P_d' is a constant, or linear.

Several of the stronger students made quite heroic efforts to prove that the critical point was a maximizer, even with these incomplete hypotheses. I was quite impressed by some of their clever arguments (even if they were incorrect). However, in the end I decided to award full marks simply for establishing that $P_d(q) = C'(q) - q \cdot P_d'(q)$ defines a critical point. □

7. Suppose Bob's monopoly was split into many competing firms, and the combined production capacity of all these firms was equal to the production capacity of Bob's original monopoly. Let P_s be the inverse function of the supply function $\overline{Q}_s$ from question #5 (that is: if $\overline{Q}_s(p) = q$, then $P_s(q) = p$). Using an argument similar to question #3, one can show that $P_s(q)$ is the *marginal cost of production* when the entire market is producing q units of icecream (you can just assume this). Since the market's marginal cost is equal to the marginal cost for Bob's original monopoly, we will have $P_s(q) = C'(q)$ for all $q \in \mathbb{R}_+$, where C is the cost function for Bob's monopoly from question #6.

[9] This is model of 'economic optimization' which is discussed in §4.7 of Stewart's book.



$(\frac{15}{200})$ (a) Sketch 'schematic' graphs of $P_d(q)$ and $P_s(q)$ $(= C'(q))$ as functions of q. Let (q^*, p^*) be the unique crossing point of these two curves. Explain why p^* is the competitive equilibrium price from question #5, and q^* is the quantity produced (and consumed) at this price.

Solution: If (q^*, p^*) is the crossing point, then $P_s(q^*) = p^* = P_d(q^*)$, which means that $\overline{Q}_s(p^*) = q^* = \overline{Q}_d(p^*)$ —in other words, p^* is the competitive equilibrium price from question #5.

⑤ Many students gave some vague intuitive/verbal argument that the intersection point corresponded to the competitive equilibrium in question #5, but they provided no mathematical justification for this claim. This is not sufficient. □

$(\frac{15}{200})$ (b) For all $q \in \mathbb{R}_+$, define $M(q) := C'(q) - q \cdot P_d'(q)$. Explain why $M(q) > C'(q)$ for all $q \in \mathbb{R}_+$.

Solution: We have $P_d'(q) < 0$ for all $q \in \mathbb{R}_+$, because P_d is a decreasing function, because it is the inverse of $\overline{Q}_d$, which we know is decreasing by question #2. Thus, $-q \cdot P_d'(q) > 0$ for all $q \in \mathbb{R}_+$. Thus, $C'(q) - q \cdot P_d'(q) > C'(q)$. □

$(\frac{15}{200})$ (c) Sketch a 'schematic' curve for M on your graph from part (a). Let (q^M, p^M) be the unique crossing point of the curve M with the curve P_d. Explain why q^M is the quantity produced by Bob's monopoly in question #6 (and p^M is the unit price which Bob charges).

Solution: If (q^M, p^M) is the crossing point, then $P_d(q^M) = M(q^M)$, which means $P_d(q) = C'(q) - q \cdot P_d'(q)$, which means q^M is the monopoly price from question #6.

⑥ Many students gave some vague intuitive/verbal argument that the intersection point corresponded to the monopolistic equilibrium in question #6, but they provided no mathematical justification for this claim. This is not sufficient. □

$(\frac{5}{200})$ (d) Conclude: assuming they have equal production capacity, a profit-maximizing monopoly will produce a *smaller* quantity of icecream than a competitive market, and will charge a *higher* price per unit sold.

Solution: It is clear from the picture that $q^M < q^*$ and $p^M > p^*$.

⑦ Many students gave some vague verbal argument about how 'monopolies are bad', making reference to 'anticompetitive behaviour', 'abuse of market power', or invoking common-sense intuitions about human psychology. This is not what I wanted. All I wanted was for you to look at the picture you had drawn, observe that (q^M, p^M) is northwest of (q^*, p^*), and draw the obvious conclusions. □

Other remarks. (a) The pattern with mistakes ⑤, ⑥, and ⑦ is that people thought I wanted an intuitive, common-sense 'economic' argument, when what I really wanted was a rigorous 'mathematical' argument. Since this is a mathematics course, it should have been obvious that I wanted the latter, not the former. Furthermore, one of my goals was to teach you the basic methodology of mathematical economics, which goes something like this:

1. Encode the economic situation in a precise mathematical model.

2. Use rigorous mathematical analysis to draw mathematical conclusions about your model.

3. Then translate these mathematical conclusions back into economic conclusions.

4. Do not mix up Step 2 and Step 3. That is: do not invoke 'economic intuition' in the middle of what is supposed to be a 'mathematical' analysis.

Now, someone might ask, 'Since these questions are ultimately about real-life economic situations, what's wrong with using our common-sense economic intuitions to answer them?' This is a good question. There is nothing wrong with using economic intuition at the appropriate moments. For example, you must use economic intuition to formulate the model in the first place (in Step 1), and also to determine whether the output of the model is realistic or if it has totally gone off the rails (in Step 3). However, as much as possible, you should proceed by rigorous mathematics, not intuition. The reason is that your intuitions can easily lead you astray. Scientific history is full of statements were believed correct because of 'common sense intuition' but which later turned out to be *wrong* for some subtle mathematical reason.

Everything I have said so far is true for *any* area of mathematical science (e.g. just substitute 'physics' for 'economics' in the above paragraphs). However, 'intuition' can be particularly pernicious in economics, because here 'intuition' is often a disguise for 'ideology'. A lot of 'common-sense' economic arguments are really just right-wing or left-wing rhetoric. As much as possible, economic reasoning should not devolve into a 'clash of ideologies' —it should try to be a rigorous, scientific (preferably mathematical) analysis.

(b) Question #7(d) shows that monopolies are bad for consumers. Now, someone might argue that the financial *gains* for the monopoly owners offset the welfare loss for consumers. However, one can prove: even if one *includes* the financial gains for the monopoly in the analysis, 'society as a whole' is worse off with the monopoly than it would be in a competitive market. (To do this, you must first define mathematical measures of the 'total consumer welfare' and 'total producer welfare' generated in a particular market; you then add these quantities to estimate 'total social welfare'.

You can then compare the 'total social welfare' of a monopoly with the 'total social welfare' of a competitive market. This kind of analysis is an example of *welfare economics*).

(c) The 'perfectly competitive' market in questions #3-#5 only applies to 'homogeneous' commodities, where there is no variation in quality, and producers compete only on the basis of price (e.g. gasoline, steel, etc.). It is *not* a good description of most 'consumer goods' (e.g. ice cream) where producers also compete by 'differentiating' their products (e.g. by introducing novel flavours, inventing new dessert products, offering different 'customer experiences' in their stores, etc.) Such a market is best described as *monopolistic competition*; the outcome will be somewhere between the extremes of questions #5 and #6. However, to mathematically study monopolistic competition, we must simultaneously model several interacting markets; this is beyond the scope of this course.