Math 1100 — Calculus, Fall Term Test — 2009-11-10

(10) 1. Compute $\lim_{x\to\infty} \dfrac{\sin(x)(x^2+1)}{x^3-4}$.

Solution: Let $f(x) := \dfrac{-(x^2+1)}{x^3-4}$, let $g(x) := \dfrac{\sin(x)(x^2+1)}{x^3-4}$, and let $h(x) := \dfrac{x^2+1}{x^3-4}$. Then

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{x^2+1}{x^3-4} = \lim_{x\to\infty} \frac{\frac{1}{x^3}(x^2+1)}{\frac{1}{x^3}(x^3-4)} = \lim_{x\to\infty} \frac{x^{-1}+\frac{1}{x^3}}{1-\frac{4}{x^3}}$$

$$= \frac{\lim_{x\to\infty} x^{-1}+\frac{1}{x^3}}{\lim_{x\to\infty} 1-\frac{4}{x^3}} = \frac{0+0}{1-0} = \frac{0}{1} = \boxed{0.}$$

Meanwhile, $f(x) = -h(x)$, so $\lim_{x\to\infty} f(x) = -\lim_{x\to\infty} h(x) = 0$.

Finally, for all $x \in \mathbb{R}_+$ we have $f(x) \leq g(x) \leq h(x)$, because $-1 \leq \sin(x) \leq 1$. Thus, the Squeeze Theorem implies that $\lim_{x\to\infty} g(x) = \boxed{0.}$ □

(15) 2. Let $f(x) = \sqrt{3+7x}$. Use the 'limit' definition of the derivative to show that

$$f'(x) = \frac{7}{2\sqrt{3+7x}}.$$

Solution:

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{\sqrt{3+7x}-\sqrt{3+7a}}{x-a}$$

$$= \lim_{x\to a} \left(\frac{\sqrt{3+7x}-\sqrt{3+7a}}{x-a}\right)\left(\frac{\sqrt{3+7x}+\sqrt{3+7a}}{\sqrt{3+7x}+\sqrt{3+7a}}\right)$$

$$= \lim_{x\to a} \frac{(\sqrt{3+7x})^2-(\sqrt{3+7a})^2}{(x-a)\left(\sqrt{3+7x}+\sqrt{3+7a}\right)} = \lim_{x\to a} \frac{3+7x-(3+7a)}{(x-a)\left(\sqrt{3+7x}+\sqrt{3+7a}\right)}$$

$$= \lim_{x\to a} \frac{7(x-a)}{(x-a)\left(\sqrt{3+7x}+\sqrt{3+7a}\right)} = \lim_{x\to a} \frac{7}{\sqrt{3+7x}+\sqrt{3+7a}}$$

$$= \frac{7}{\sqrt{3+7a}+\sqrt{3+7a}} = \boxed{\frac{7}{2\sqrt{3+7a}}}$$

□

3. You can use any 'differentiation rules' you want to answer the following questions (e.g. Leibniz product rule, chain rule, implicit differentiation, etc.):

(10) (a) Prove that $\cot'(x) = -\csc(x)^2$, for all $x \in \mathbb{R}$ where it is defined.

Solution: $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$. Thus, the quotient rule says that

$$\begin{aligned}\cot'(x) &= \frac{\cos'(x)\sin(x) - \cos(x)\sin'(x)}{\sin(x)^2} = \frac{-\sin(x)\sin(x) - \cos(x)\cos(x)}{\sin(x)^2} \\ &= \frac{-(\sin(x)^2 + \cos(x)^2)}{\sin(x)^2} = \frac{-1}{\sin(x)^2} = -\csc(x)^2.\end{aligned}$$

□

(15) (b) Prove that $\arccos'(x) = \dfrac{-1}{\sqrt{1-x^2}}$, for all $x \in \mathbb{R}$ where it is defined.

Solution: Let $y = \arccos(x)$; then $x = \cos(y)$. Now,

$$\cos'(y) = -\sin(y) = -\sqrt{1-\cos^2(y)} = -\sqrt{1-x^2}$$

Thus, the Inverse Function Theorem says

$$\arccos'(x) = \frac{1}{\cos'(y)} = \frac{1}{-\sqrt{1-x^2}},$$

as desired. □

4. Differentiate the following functions:

(10) (a) $f(x) = \sqrt{\cot(x)}$.

Solution: $f = h \circ \cot$, where $h(y) = \sqrt{y}$. Now, $h'(y) = \dfrac{1}{2\sqrt{y}}$, and $\cot'(x) = -\csc^2(x)$. Thus, the Chain Rule says

$$f'(x) = h'[\cot(x)] \cdot \cot'(x) = \boxed{\frac{-\csc(x)^2}{2\sqrt{\cot(x)}}}.$$

□

(10) (b) $g(x) = e^{\sqrt{\cot(x)}}$.

Solution: $g = \exp \circ f$, where f is from the previous question. We have $f'(x) = \dfrac{-\csc(x)^2}{2\sqrt{\cot(x)}}$. Thus, the Chain Rule says

$$g'(x) = \exp'(f(x)) \cdot f'(x) = \exp(f(x)) \cdot f'(x) = \boxed{\frac{-e^{\sqrt{\cot(x)}} \cdot \csc(x)^2}{2\sqrt{\cot(x)}}}.$$

□

(15) (c) $f(x) = \sqrt{x} \cdot e^{(x^2)} \cdot (x^2+1)^{10} \cdot \tan(x)$.

Solution: Let $L(x) := \ln[f(x)]$. Then $L'(x) = f'(x)/f(x)$, so $f'(x) = f(x) \cdot L'(x)$. But

$$\begin{aligned} L(x) &= \ln\left(\sqrt{x} \cdot e^{x^2} \cdot (x^2+1)^{10} \cdot \tan(x)\right) \\ &= \ln(\sqrt{x}) + \ln(e^{x^2}) + \ln\left[(x^2+1)^{10}\right] + \ln[\tan(x)] \\ &= \frac{1}{2}\ln(x) + x^2 + 10\ln(x^2+1) + \ln[\tan(x)] \\ \text{so } L'(x) &= \frac{1}{2x} + 2x + 10\frac{2x}{x^2+1} + \frac{\sec^2(x)}{\tan(x)}; \\ \text{so } f'(x) &= f(x) \cdot L'(x) = \boxed{\sqrt{x} \cdot e^{x^2} \cdot (x^2+1)^{10} \cdot \tan(x) \cdot \left(\frac{1}{2x} + 2x + \frac{20x}{x^2+1} + \frac{\sec^2(x)}{\tan(x)}\right)}. \end{aligned}$$

□

(15) 5. Let $\mathcal{C}$ be the set of all points (x, y) in the plane satisfying the equation

$$\sqrt{x+y} = 1 + x^2 \cdot y^2.$$

Find a formula for the tangent slope at any point (x, y) in $\mathcal{C}$.

Solution: We apply implicit differentiation. Suppose there is some function $\gamma : \mathcal{X} \longrightarrow \mathbb{R}$ such that

$$\sqrt{x + \gamma(x)} = 1 + x^2 \cdot \gamma(x)^2, \qquad \text{for all } x \in \mathcal{X}.$$

Differentiating both sides of this equation, we get:

$$\frac{1 + \gamma'(x)}{2\sqrt{x + \gamma(x)}} = 2x \cdot \gamma(x)^2 + 2x^2\gamma(x)\gamma'(x), \qquad \text{for all } x \in \mathcal{X}.$$

Now we simplify and isolate $\gamma'(x)$. We have:

$$\frac{\gamma'(x)}{2\sqrt{x + \gamma(x)}} - 2x^2\gamma(x)\gamma'(x) = 2x \cdot \gamma(x)^2 - \frac{1}{2\sqrt{x + \gamma(x)}}, \qquad \text{for all } x \in \mathcal{X},$$

and hence

$$\left(\frac{1}{2\sqrt{x + \gamma(x)}} - 2x^2\gamma(x)\right) \cdot \gamma'(x) = 2x \cdot \gamma(x)^2 - \frac{1}{2\sqrt{x + \gamma(x)}}, \qquad \text{for all } x \in \mathcal{X},$$

and hence

$$\gamma'(x) = \frac{2x \cdot \gamma(x)^2 - \frac{1}{2\sqrt{x+\gamma(x)}}}{\frac{1}{2\sqrt{x+\gamma(x)}} - 2x^2\gamma(x)}, \qquad \text{for all } x \in \mathcal{X}.$$

If $y = \gamma(x)$, this simplifies to:

$$\text{slope}(x, y) = \frac{2x \cdot y^2 - \frac{1}{2\sqrt{x+y}}}{\left(\frac{1}{2\sqrt{x+y}} - 2x^2 y\right)} = \boxed{\frac{4x \cdot y^2\sqrt{x+y} - 1}{1 - 4x^2 y\sqrt{x+y}}}, \qquad \text{for all } x \in \mathcal{X}.$$

□