MATH 1101 2009
Assignment 1
Due November 3, 2009

5 points for each problem. Show all your work.

1. Find the domain of the function and a formula for the inverse of the function.

(a)

$$y = \ln\left(x^3 - 3\right)$$

Solution: To find the domain, we need to solve the inequality

$$\begin{aligned} x^3 - 3 &> 0 \\ &\Leftrightarrow x > \sqrt[3]{3}. \end{aligned}$$

The domain is $\left(\sqrt[3]{3}, \infty\right)$.
To find the inverse:

$$\begin{aligned} y &= \ln\left(x^3 - 3\right) \\ &\Leftrightarrow e^y = x^3 - 3 \\ &\Leftrightarrow x^3 = e^y + 3 \\ &\Leftrightarrow x = \sqrt[3]{e^y + 3}. \end{aligned}$$

The inverse is $y = \sqrt[3]{e^x + 3}$. □

(b)

$$y = \frac{e^x}{1 + 2e^x}$$

Solution: Since $1 + 2e^x > 1$, the domain is $(-\infty, \infty)$.
To find the inverse:

$$\begin{aligned} y &= \frac{e^x}{1 + 2e^x} \\ &\Leftrightarrow (1 + 2e^x)\, y = e^x \\ &\Leftrightarrow y + 2e^x y = e^x \\ &\Leftrightarrow 2e^x y - e^x = -y \\ &\Leftrightarrow (2y - 1)\, e^x = -y \\ &\Leftrightarrow e^x = \frac{-y}{2y - 1} \\ &\Leftrightarrow x = \ln\left(\frac{-y}{2y - 1}\right). \end{aligned}$$

The inverse is $y = \ln\left(\frac{-x}{2x-1}\right)$. □

2. Use the $\epsilon - \delta$ definition to prove

$$\lim_{x \to 3} \frac{1}{2}x - 1 = \frac{1}{2}$$

Proof: We want to show that for any $\epsilon > 0$, there is a $\delta > 0$ such that whenever $|x-3| < \delta$, $\left|\left(\frac{1}{2}x - 1\right) - \frac{1}{2}\right| < \epsilon$. Since

$$\left|\left(\frac{1}{2}x - 1\right) - \frac{1}{2}\right| < \epsilon$$

$$\begin{aligned}
\Leftrightarrow &\quad \left|\frac{1}{2}x - \frac{3}{2}\right| < \epsilon \\
\Leftrightarrow &\quad \frac{1}{2}|x-3| < \epsilon \\
\Leftrightarrow &\quad |x-3| < 2\epsilon
\end{aligned}$$

we choose $\delta = 2\epsilon$. Therefore whenever

$$|x-3| < \delta$$

$$\begin{aligned}
\Leftrightarrow &\quad |x-3| < 2\epsilon \\
\Leftrightarrow &\quad \frac{1}{2}|x-3| < \epsilon \\
\Leftrightarrow &\quad \left|\frac{1}{2}x - \frac{3}{2}\right| < \epsilon \\
\Leftrightarrow &\quad \left|\left(\frac{1}{2}x - 1\right) - \frac{1}{2}\right| < \epsilon.
\end{aligned}$$

□

3. Use the Intermediate Value Theorem to prove that the equation

$$\ln(1+x) = 3 - 2x$$

has a real root and use your calculator to find an interval of length at most 0.01 that contains a root.

Solution: Let $f(x) = \ln(1+x) - (3-2x) = \ln(1+x) - 3 + 2x$. We have $f(0) = -3 < 0$ and $f(2) = \ln 3 + 1 > 0$. By the Intermediate Value Theorem there is a c in $(0,2)$ such that $f(c) = 0$. c is a root of the original equation.

Since $f(1) = \ln 2 - 1 = -0.306\,85 < 0$, there is a root of the equation in $(1,2)$.

Since $f(1.5) = \ln(2.5) = 0.916\,29 > 0$, there is a root of the equation in $(1,1.5)$.

Since $f(1.25) = \ln(2.25) - 0.5 = 0.310\,93 > 0$, there is a root of the equation in $(1,1.25)$.

Since $f(1.125) = \ln(2.125) - 0.75 = 3.\,771\,8 \times 10^{-3} > 0$, there is a root of the equation in $(1,1.125)$.

Since $f(1.1125) = \ln(2.1125) - 0.875 = -0.127\,13 < 0$, there is a root of the equation in $(1.1125, 1.125)$.

Since $f(1.\,11875) = \ln(2.11875) - 0.7625 = -1.\,167\,4 \times 10^{-2} < 0$, there is a root of the equation in $(1.11875, 1.125)$. We have $1.125 - 1.11875 = 0.006\,25 < 0.01$. $\square$

4. Use limits to find all the asymptotes of the curve

$$y = \frac{x^2 - 3x + 2}{(x-5)\sqrt{x^2 - 4}}$$

Solution: The domain of the function is $(-\infty, -2) \cup (2, 5) \cup (5, \infty)$. To find the vertical asymptotes, we consider the limits of y as $x \to -2^-$, $x \to 2^+$ and $x \to 5^-$, $x \to 5^+$.

$$\begin{aligned} & \lim_{x \to -2^-} \frac{x^2 - 3x + 2}{(x-5)\sqrt{x^2 - 4}} \\ = & \lim_{x \to -2^-} \frac{(x-1)(x-2)}{(x-5)\sqrt{(x+2)(x-2)}} = -\infty \end{aligned}$$

$$\begin{aligned} & \lim_{x \to 2^+} \frac{x^2 - 3x + 2}{(x-5)\sqrt{x^2 - 4}} \\ = & \lim_{x \to 2^+} \frac{(x-1)(x-2)}{(x-5)\sqrt{(x+2)(x-2)}} \\ = & \lim_{x \to 2^+} \frac{(x-1)\sqrt{x-2}}{(x-5)\sqrt{x+2}} = 0 \end{aligned}$$

$$\lim_{x \to 5^-} \frac{x^2 - 3x + 2}{(x-5)\sqrt{x^2 - 4}} = -\infty$$

$$\lim_{x \to 5^+} \frac{x^2 - 3x + 2}{(x-5)\sqrt{x^2 - 4}} = \infty$$

Therefore, the vertical asymptotes are $x = -2$ and $x = 5$.

To find the horizontal asymptotes, we calculate the limits of y as $x \to -\infty$ and $x \to \infty$.

$$\begin{aligned} & \lim_{x \to -\infty} \frac{x^2 - 3x + 2}{(x-5)\sqrt{x^2 - 4}} \\ = & \lim_{x \to -\infty} \frac{\frac{x^2 - 3x + 2}{x^2}}{\frac{(x-5)\sqrt{x^2-4}}{x^2}} \\ = & \lim_{x \to -\infty} \frac{1 - \frac{3}{x} + \frac{2}{x^2}}{\left(1 - \frac{5}{x}\right)(-1)\sqrt{1 - \frac{4}{x^2}}} = -1 \end{aligned}$$

$$
\begin{aligned}
& \lim_{x\to\infty} \frac{x^2-3x+2}{(x-5)\sqrt{x^2-4}} \\
= & \lim_{x\to\infty} \frac{\frac{x^2-3x+2}{x^2}}{\frac{(x-5)\sqrt{x^2-4}}{x^2}} \\
= & \lim_{x\to-\infty} \frac{1-\frac{3}{x}+\frac{2}{x^2}}{\left(1-\frac{5}{x}\right)\sqrt{1-\frac{4}{x^2}}} = 1
\end{aligned}
$$

The horizontal asymptotes are $y = 1$ and $y = -1$. $\square$