Mathematics 110 – Calculus of one variable
Trent University 2003-2004
SOLUTIONS TO ASSIGNMENT #2

Odds and ends

It is noted in the text that $f(x) = \sin\left(\frac{1}{x}\right)$ is not continuous at $a = 0$, from which it follows that is not differentiable at $a = 0$. By way of contrast, $g(x) = x\sin\left(\frac{1}{x}\right)$ is continuous, but not differentiable at $a = 0$. Your task, should you choose to accept it, is:

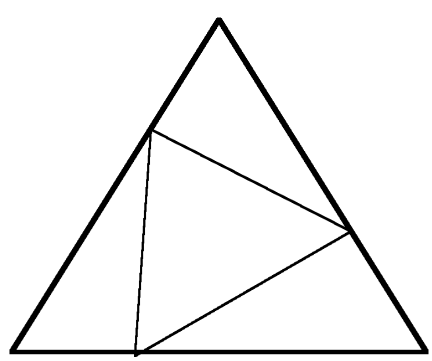
1. Check that $h(x) = x^2\sin\left(\frac{1}{x}\right)$ is differentiable (and hence continuous) at $a = 0$. *[2]*

SOLUTION. For this to make sense, we have to define $h(0)$ to be 0, since otherwise $h(x)$ wouldn't even be defined, much less continuous, at $a = 0$. Now, by the limit definition of the derivative:

$$h'(0) = \lim_{t\to 0}\frac{h(0+t)-h(0)}{t} = \lim_{t\to 0}\frac{h(t)-0}{t} = \lim_{t\to 0}\frac{t^2\sin\left(\frac{1}{t}\right)}{t} = \lim_{t\to 0} t\sin\left(\frac{1}{t}\right)$$

We will show that the last limit exists and equals 0. For any $t \neq 0$, $-1 \leq \sin\left(\frac{1}{t}\right) \leq 1$, so $-t \leq t\sin\left(\frac{1}{t}\right) \leq t$. Since $\lim_{t\to 0} -t = \lim_{t\to 0} t = 0$, it follows by the Squeeze Theorem that $\lim_{t\to 0} t\sin\left(\frac{1}{t}\right) = 0$. Hence $h(x)$ is diffrentiable at $a = 0$. ■

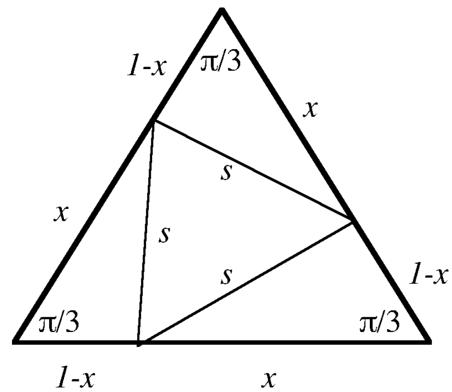
Suppose an equilateral triangle is inscribed inside an equilateral triangle with sides of length 1.



2. What is the minimum area of such an inscribed triangle? *[4]*

SOLUTION. We'll need a couple of formulas. First, the area of an equilateral triangle with sides of length s is $\frac{\sqrt{3}}{4}s^2$. (You can get this from the formula for the area of a triangle, $\frac{1}{2}\cdot\text{base}\cdot\text{height}$, by using one of the sides of the equilateral triangle as the base and then using the Pythagorean Theorem to find the height.) Second, the Law of Cosines: if an arbitrary triangle has sides of lengths a, b, and c, and C is the angle opposite the side of length c, then $c^2 = a^2 + b^2 - 2ab\cos(C)$. (This is a generalization of the Pythagorean Theorem to non-right triangles; you can look it up too!)

In the diagram of the triangles given in the problem, the vertices of the inscribed triangle divide the sides of the equilateral tringle with sides of length one into two parts. If one of these two parts has length x, the other must have length $1 - x$. We will let s be the length of a side of the inscribed triangle. Note that the interior angles of an equilateral triangle are all $\pi/3$ radians.



Using the Law of Cosines on (any) one of the three sub-triangles with sides of length s, x, and $1 - x$ gives:

$$\begin{aligned} s^2 &= x^2 + (1-x)^2 - 2x(1-x)\cos(\pi/3) \\ &= x^2 + 1 - 2x + x^2 - 2\left(x - x^2\right)\frac{1}{2} \\ &= x^2 + 1 - 2x + x^2 - x + x^2 \\ &= 3x^2 - 3x + 1 \end{aligned}$$

Note that $\cos(\pi/3) = \dfrac{1}{2}$. It follows that the area of the inscribed triangle is, in terms of x:

$$A(x) = \frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4}\left(3x^2 - 3x + 1\right)$$

Note that $0 \le x \le 1$.

We can now find the minimum value of $A(x)$ for $0 \le x \le 1$ as usual. Find the critical point(s):

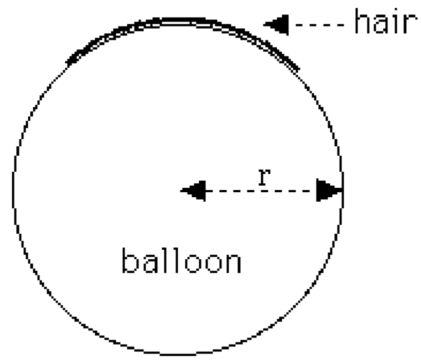
$$A'(x) = \frac{d}{dx}\frac{\sqrt{3}}{4}\left(3x^2 - 3x + 1\right) = \frac{\sqrt{3}}{4}(6x - 3) = 0 \iff x = \frac{1}{2}$$

Compare the values of $A(x)$ at the critical point and the endpoints:

$$\begin{aligned} A\left(\frac{1}{2}\right) &= \frac{\sqrt{3}}{4}\left(\frac{3}{4} - \frac{3}{2} + 1\right) = \frac{\sqrt{3}}{16} \\ A(0) &= \frac{\sqrt{3}}{4}(0 - 0 + 1) = \frac{\sqrt{3}}{4} \\ A(1) &= \frac{\sqrt{3}}{4}(3 - 3 + 1) = \frac{\sqrt{3}}{4} \end{aligned}$$

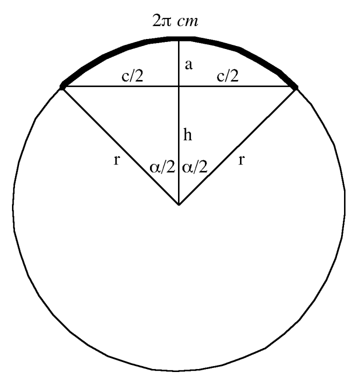
The minimum value of $A(x)$, the area of an inscribed triangle, is therefore $A\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{16}$. ∎

A hair 2π *cm* long lies as straight as possible on the surface of a spherical balloon while it is being inflated. The balloon remains spherical at all times, and the hair, which doesn't stretch or shrink, remains as straight as possible on its surface.



3. How is the radius of the balloon changing when it is 4 *cm*, if the ends of the hair are moving apart at 1 *cm/s* at that instant? *[2]*

SOLUTION. The key to understanding this set-up is that it is really only two-dimensional. If you were to cut the balloon — imagine that it wouldn't just pop! — along the hair, you would cut right through the center of the balloon. A diagram of the resulting cross-section, with various lines and angles drawn in and labelled, is given below. Note that the cross-section is just a circle with the same radius as the balloon.



We'll need some of the fundamental relationships among the various items mentioned in the diagram. First, note that, in a circle of radius r, the length of the arc subtended by an angle of α radians — that is, the length of the hair — is just $r\alpha$. (This simplicity is one of the pleasant benefits of using radians.) In our set-up, this means that

$$r\alpha = 2\pi \,.$$

This also lets us determine the angle α at the instant when $r = 4$: if $4\alpha = 2\pi$, then $\alpha = \frac{\pi}{2}$.

Second, the length c of the chord corresponding to the arc subtended by α — that is, the distance between the ends of the hair — can be computed from one the symmetric right triangles in the diagram, to get $\frac{c}{2} = r \sin\left(\frac{\alpha}{2}\right)$. Thus

$$c = 2r \sin\left(\frac{\alpha}{2}\right) \,,$$

and, at the instant when $r = 4$ and $\alpha = \frac{\pi}{2}$, $c = 2 \cdot 4 \cdot \sin\left(\frac{\pi}{4}\right) = 8 \cdot \frac{1}{\sqrt{2}} = 4\sqrt{2}$.

Third, the distance a between the midpoint of the hair and the chord — that is, line between the two ends of the hair — can be obtained by subtracting the radius of the balloon from the height of the triangle whose base is the chord and whose tip is the centre of the balloon. Using one the symmetric right triangles again gives us $r^2 = h^2 + \left(\frac{c}{2}\right)^2$, so $h = \sqrt{r^2 - \frac{c^2}{4}}$. Thus

$$a = r - h = r - \sqrt{r^2 - \frac{c^2}{4}}.$$

Now, what is $\frac{dr}{dt}$ when $r = 4$ cm, if $\frac{dc}{dt} = 1$ cm/s at the same instant?

We are told that $\frac{dc}{dt} = 1$ and need to find $\frac{dr}{dt}$. Since $r\alpha = 2\pi$, we have that $\alpha = \frac{2\pi}{r}$. So we can express c in terms of r alone,

$$c = 2r \sin\left(\frac{2\pi}{2r}\right) = 2r \sin\left(\frac{\pi}{r}\right),$$

and differentiate away with respect to t on both sides:

$$\begin{aligned}
\frac{dc}{dt} &= 2\frac{dr}{dt} \cdot \sin\left(\frac{\pi}{r}\right) + 2r \cdot \cos\left(\frac{\pi}{r}\right) \cdot \frac{d}{dt}\left(\frac{\pi}{r}\right) \\
&= 2\frac{dr}{dt} \cdot \sin\left(\frac{\pi}{r}\right) + 2r \cdot \cos\left(\frac{\pi}{r}\right) \cdot \frac{-\pi}{r^2} \cdot \frac{dr}{dt} \\
&= 2\frac{dr}{dt} \cdot \sin\left(\frac{\pi}{r}\right) - \frac{2\pi}{r} \cdot \cos\left(\frac{\pi}{r}\right) \cdot \frac{dr}{dt} \\
&= \frac{dr}{dt} \cdot \left[2\sin\left(\frac{\pi}{r}\right) - \frac{2\pi}{r} \cdot \cos\left(\frac{\pi}{r}\right)\right].
\end{aligned}$$

Since $r = 4$ at the instant in question, this gives

$$\begin{aligned}
1 &= \frac{dr}{dt} \cdot \left[2\sin\left(\frac{\pi}{4}\right) - \frac{2\pi}{4} \cdot \cos\left(\frac{\pi}{4}\right)\right] \\
&= \frac{dr}{dt} \cdot \left[\frac{2}{\sqrt{2}} - \frac{\pi}{2} \cdot \frac{1}{\sqrt{2}}\right] \\
&= \frac{dr}{dt} \cdot \left[\frac{4-\pi}{2\sqrt{2}}\right].
\end{aligned}$$

It follows that the radius is changing at a rate of

$$\frac{dr}{dt} = \frac{2\sqrt{2}}{4-\pi}.$$

That amounts to roughly 3.3. Thus, at the instant in question, the radius of the balloon is growing — note that $\frac{dr}{dt}$ is positive! — at a rate of about 3.3 cm/s. ■

4. At the same instant, how quickly is the midpoint of the hair aproaching the line between the two ends? *[2]*

Solution. At the same instant, how quickly is the midpoint of the hair aproaching the line between the two ends?

We know $\frac{dr}{dt}$ at this instant from question **3**, and we are given $\frac{dc}{dt}$, so all we have to do is differentiate away in

$$a = r - h = r - \sqrt{r^2 - \frac{c^2}{4}}\,.$$

Then

$$\begin{aligned}
\frac{da}{dt} &= \frac{dr}{dt} - \frac{d}{dt}\left(\sqrt{r^2 - \frac{c^2}{4}}\right) \\
&= \frac{dr}{dt} - \frac{1}{2\sqrt{r^2 - \frac{c^2}{4}}} \cdot \frac{d}{dt}\left(r^2 - \frac{c^2}{4}\right) \\
&= \frac{dr}{dt} - \frac{1}{2\sqrt{r^2 - \frac{c^2}{4}}} \cdot \left(2r \cdot \frac{dr}{dt} - \frac{2c}{4} \cdot \frac{dc}{dt}\right) \\
&= \frac{dr}{dt} - \frac{2r \cdot \frac{dr}{dt} - \frac{c}{2} \cdot \frac{dc}{dt}}{2\sqrt{r^2 - \frac{c^2}{4}}}\,,
\end{aligned}$$

so, at the instant in question,

$$\begin{aligned}
\frac{da}{dt} &= \frac{2\sqrt{2}}{4-\pi} - \frac{2 \cdot 4 \cdot \frac{2\sqrt{2}}{4-\pi} - \frac{4\sqrt{2}}{2} \cdot 1}{2\sqrt{4^2 - \frac{(4\sqrt{2})^2}{4}}} \\
&= \frac{2\sqrt{2}}{4-\pi} - \frac{\frac{16\sqrt{2}}{4-\pi} - 2\sqrt{2}}{2\sqrt{16-8}} \\
&= \frac{2\sqrt{2}}{4-\pi} - \frac{2\sqrt{2} \cdot \left(\frac{8}{4-\pi} - 1\right)}{4\sqrt{2}} \\
&= \frac{2\sqrt{2}}{4-\pi} - \frac{1}{2} \cdot \left(\frac{4+\pi}{4-\pi}\right) \\
&= \frac{2\sqrt{2}}{4-\pi} - \frac{4+\pi}{2(4-\pi)} \\
&= \frac{4\sqrt{2} - 4 - \pi}{8 - 2\pi}\,.
\end{aligned}$$

That amounts to roughly -0.9. This means that the middle of the hair is getting closer — note the sign! — to the line joining the ends at a speed of about 0.9 cm/s. ■