

# Mathematics 110 – Calculus of one variable

TRENT UNIVERSITY 2003-2004

ASSIGNMENT #6

Due: March 31, 2004

**Instructions:** Do *one* of Parts **1** and **2**.

## Part 1

You may assume that the following result is true.

**Theorem** (Stolz - Cesaro) Let  $(a_n)_{n \geq 1}$  and  $(b_n)_{n \geq 1}$  be two sequences of real numbers such that  $(b_n)_{n \geq 1}$  is increasing,  $b_n > 0$  for all  $n \geq 1$ ,  $\lim_{n \rightarrow \infty} b_n = \infty$ , and

$$\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$$

exists or is equal to  $\pm\infty$ . Then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$$

1. Let  $p \in \mathbb{R}$ ,  $p \neq -1$ . Using the Stolz-Cesaro Theorem, compute the limit:

$$\lim_{n \rightarrow \infty} \frac{1^p + 2^p + \dots + n^p}{n^{p+1}} \quad [4]$$

2. Let  $(c_n)_{n \geq 1}$  be a sequence of positive real numbers. Use the Stolz-Cesaro Theorem to show that if

$$\lim_{n \rightarrow \infty} \frac{c_{n+1}}{c_n}$$

exists or is  $\pm\infty$ , then

$$\lim_{n \rightarrow \infty} \sqrt[n]{c_n} = \lim_{n \rightarrow \infty} \frac{c_{n+1}}{c_n} \quad [4]$$

3. Use **2** to compute:

$$\lim_{n \rightarrow \infty} \sqrt[n]{n!} \quad [2]$$

## Part 2

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be the function defined by  $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$ .

1. Using mathematical induction, show that

$$f^{(n)}(x) = P_{3n} \left( \frac{1}{x} \right) e^{-\frac{1}{x^2}}$$

for all  $x \neq 0$  and all  $n \geq 1$ , where  $P_{3n}$  is a polynomial of degree  $3n$  and  $f^{(n)}$  denotes the  $n$ -th derivative of  $f$ . [4]

2. Show that  $f^{(n)}(0) = 0$  for all  $n \geq 1$ . [4]
3. For what  $x$  is  $f(x)$  equal to its Maclaurin series (that is, its Taylor series at 0)? Justify your answer. [2]

## Glossa\*

**Time goes by, time comes along,  
All is old and all is new;  
What is right and what is wrong,  
You must think and ask of you;  
Have no hope and have no fear,  
Waves that rise can never hold;  
If they urge or if they cheer,  
You remain aloof and cold.**

To our sight a lot will glisten,  
Many sounds will reach our ear;  
Who could take the time to listen  
And remember all we hear?  
Keep aside from all that patter,  
Seek yourself, far from the throng  
When with loud and idle clatter  
**Time goes by, time comes along.**

Nor forget the tongue of reason  
Or its even scales depress  
When the moment, changing season,  
Wears the mask of happiness -  
It is born of reason's slumber  
And may last a wink as true:  
For the one who knows its number  
**All is old and all is new.**

Be as to a play, spectator,  
As the world unfolds before:  
You will know the heart of matter  
Should they act two parts or four;  
When they cry or tear asunder  
From your seat enjoy along  
And you'll learn from art to wonder  
**What is right and what is wrong.**

Past and future, ever blending,  
Are the twin sides of same page:  
New start will begin with ending  
When you know to learn from age;  
All that was or be tomorrow  
We have in the present, too;  
But what's vain and futile sorrow  
**You must think and ask of you;**

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\* Fixed Latin form of lyric verse, generally used for poems of philosophic character, wherein each line of the first stanza is commented in subsequent stanzas ending with that line. The last stanza repeats the first in reverse order.

For the living cannot sever  
From the means we've always had:  
Now, as years ago, and ever,  
Men are happy or are sad:  
Other masks, same play repeated;  
Diff'rent tongues, same words to hear;  
Of your dreams so often cheated,  
**Have no hope and have no fear.**

Hope not when the villains cluster  
By success and glory drawn:  
Fools with perfect lack of luster  
Will outshine Hyperion!  
Fear it not, they'll push each other  
To reach higher in the fold,  
Do not side with them as brother,  
**Waves that rise can never hold.**

Sounds of siren songs call steady  
Toward golden nets, astray;  
Life attracts you into eddies  
To change actors in the play;  
Steal aside from crowd and bustle,  
Do not look, seem not to hear  
From your path, away from hustle  
**If they urge or if they cheer;**

If they reach for you, go faster,  
Hold your tongue when slanders yell;  
Your advice they cannot master,  
Don't you know their measure well?  
Let them talk and let them chatter,  
Let all go past, young and old;  
Unattached to man or matter,  
**You remain aloof and cold.**

**You remain aloof and cold**  
**If they urge or if they cheer;**  
**Waves that rise can never hold,**  
**Have no hope and have no fear;**  
**You must think and ask of you**  
**What is right and what is wrong;**  
**All is old and all is new,**  
**Time goes by, time comes along.**

*By Mihai Eminescu (1850 - 1889). Translation from Romanian by A.G. Sahlean.  
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