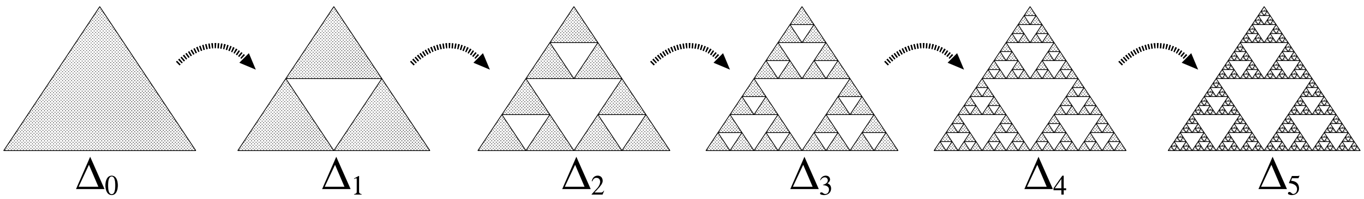

Figure 1: The Sierpinski triangle

Math 110 —Assignment #9

Due: Wednesday March 26th

- *Justify your answers.* Show all steps in your computations.
- Please indicate your final answer by putting a $\boxed{\text{box}}$ around it.
- Please write neatly and legibly. *Illegible answers will not be graded.*
- **Section A:** When finished, please place your assignment under Stefan's door.
  Section B: When finished, please place your assignment in slot marked MATH 110 in the big white box outside the Math Department Office in Lady Eaton College.

1. The **Sierpinski Triangle** is constructed as follows (see Figure 1)

   Step 0: Begin with an equilateral triangle Δ_0,

   Step 1: Remove an upside-down triangle from the middle of Δ_0, leaving a region Δ_1 which looks like three smaller triangles (each half the side-length of Δ_0).

   Step 2: Remove an upside-down triangle from each of these smaller triangles, leaving a region Δ_2, which is a union of nine triangles (each one quarter the side-length of Δ_0.

   If we iterate this process an infinite number of times, the remaining set, Δ_∞, is the Sierpinski triangle.

   Let A_n be the area of Δ_n. Assume for simplicity that $A_0 = 1$.

   (a) Show that $A_1 = \frac{3}{4}$. (**Hint:** *Show that the central triangle which we remove has an area of* $\frac{1}{4}$)

   Solution: Let T_0 be the central triangle. Observe that Δ_0 is made of four copies of T_0. Thus, $1 = area(\Delta_0) = 4 \cdot area(T_0)$, so $area(T_0) = \frac{1}{4}$.

   (b) Show that $A_{n+1} = \frac{3}{4} A_n$ for all $n > 1$. Conclude that $A_n = \left(\frac{3}{4}\right)^n$ for all $n \in \mathbb{N}$.

   Solution: Δ_n consists of 3^n tiny triangular components, each of which looks like Δ_0. To obtain Δ_{n+1}, we replace each of these triangles with a tiny copy of Δ_1. Thus, by **(a)**, we reduce the area of each triangular component by a factor of $\frac{3}{4}$. Since we do this to every component, we also reduce the area of Δ_n by a factor of $\frac{3}{4}$.

   Suppose (by induction) that $A_{n-1} = \left(\frac{3}{4}\right)^{n-1}$. Then $A_n = \frac{3}{4} A_{n-1} = \frac{3}{4} \cdot \left(\frac{3}{4}\right)^{n-1} = \left(\frac{3}{4}\right)^n$.

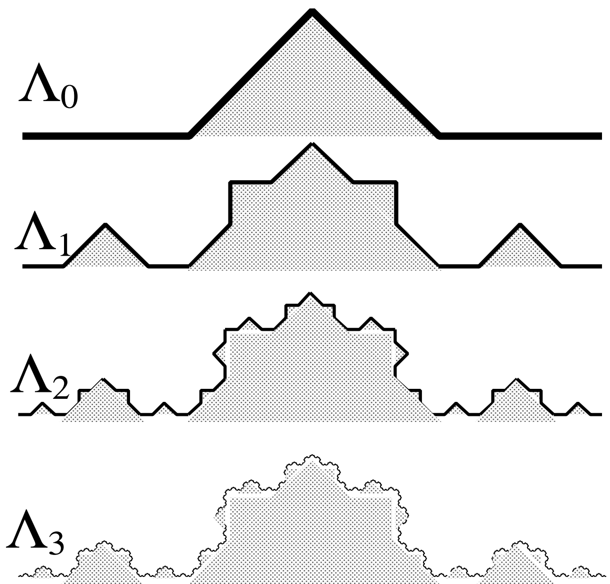

Figure 2: The Continent of Koch

(c) Compute A_∞, the area of Δ_∞.

Solution: $A_\infty = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \left(\frac{3}{4}\right)^n = \boxed{0.}$

2. The **Continent of Koch** is constructed as follows (see Figure 2):

Step 0: Begin with four equal line segments, each of length 1, arranged as Λ_0.

Step 1: Replace each of these line segments with a $\frac{1}{3}$-scale copy of Λ_0, to obtain Λ_1. Observe that Λ_1 consists of 16 line segments, each of length $\frac{1}{3}$.

Step 2: Replace each of these 16 line segments with a $\frac{1}{9}$-scale copy of Λ_0, to obtain Λ_2. Observe that Λ_2 consists of 64 line segments, each of length $\frac{1}{9}$.

Iterate this process an infinite number of times. The curve you obtain is Λ_∞, the coastline of the Continent.

(a) Let P_n be the length of Λ_n. Thus, $P_0 = 4$, because Λ_0 consists of four line segments of length 1. Show that $P_1 = \frac{16}{3}$.

Solution: Λ_1 consists of 16 line segments, each of length $\frac{1}{3}$, for a total length of $16 \cdot \frac{1}{3} = \frac{16}{3}$.

(b) Show that $P_{n+1} = \frac{4}{3} P_n$ for all $n > 1$. Conclude that $P_n = 4\left(\frac{4}{3}\right)^n$ for all $n \in \mathbb{N}$.

Solution: Λ_n consists of 4^{n+1} tiny line segments. To obtain Λ_{n+1}, we replace every line segment in Λ_n with four smaller line segments, each one third as long. Thus, we effectively increase the length of each line segment by a factor of $\frac{4}{3}$. This means we increase the total length of Λ_n by a factor of $\frac{4}{3}$.

Assume inductively that $P_{n-1} = \left(\frac{4}{3}\right)^{n-1}$. Then $P_n = \frac{4}{3} P_{n-1} = \frac{4}{3} \cdot 4 \left(\frac{4}{3}\right)^{n-1} = 4\left(\frac{4}{3}\right)^n$.

(c) Let P_∞ be the length of Λ_∞; compute P_∞, the length of the coastline of Koch.

Solution: $P_\infty = \lim_{n\to\infty} P_n = \lim_{n\to\infty} \left(\frac{4}{3}\right)^n = \boxed{\infty.}$

(d) Let A_n be the area under the curve Λ_n (ie. the shaded region in Figure 2). Assume $A_0 = \frac{\sqrt{3}}{4}$. Show that $A_1 = A_0 + \frac{4}{9}A_0$.

Solution: We obtain Λ_1 by adding to Λ_0 four smaller copies of itself, each one third in size. Thus, each of these four copies has an area of $\frac{1}{9}A_0$, so the total area of Λ_1 is $A_0 + \frac{4}{9}A_0$.

(e) Show that $A_n = A_{n-1} + \left(\frac{4}{9}\right)^n A_0$. Conclude that $A_n = A_0 \cdot \sum_{i=0}^{n} \left(\frac{4}{9}\right)^i$.

Solution: Λ_{n-1} consists of 4^n line segments, each of length $\frac{1}{3^{n-1}}$. We obtain Λ_{n+1} by attaching a tiny copy of Λ_0 to each of these. Each tiny copy is $\frac{1}{3^n}$ times the size of Λ_0, so its area is $\frac{1}{9^n}A_0$. We are attaching a total of 4^n copies, for a total additional area of $4\left(\frac{4}{9}\right)^n A_0$. Thus, $A_n = A_{n-1} + 4\left(\frac{4}{9}\right)^n A_0$.

Assume inductively that $A_{n-1} = \sum_{i=0}^{n-1} \left(\frac{4}{9}\right)^i$. Then $A_n = A_{n-1} + \left(\frac{4}{9}\right)^n A_0 = \sum_{i=0}^{n-1} \left(\frac{4}{9}\right)^i + \left(\frac{4}{9}\right)^n A_0 = \sum_{i=0}^{n} \left(\frac{4}{9}\right)^i$.

(f) Compute A_∞, the area of the Continent of Koch.

Solution: $A_\infty = \lim_{n\to\infty} A_n = \lim_{n\to\infty} A_0 \sum_{i=0}^{n} \left(\frac{4}{9}\right)^i = A_0 \sum_{i=0}^{\infty} \left(\frac{4}{9}\right)^i = A_0 \cdot \frac{1}{1-\frac{4}{9}} = A_0 \cdot \frac{1}{5/9} = A_0 \cdot \frac{9}{5} = \frac{9}{5} \cdot \frac{\sqrt{3}}{4} = \boxed{\frac{9\sqrt{3}}{20}.}$