Mathematics 4790H – Analysis II: Topology and Measure
TRENT UNIVERSITY, Winter 2025

TAKE-HOME FINAL EXAMINATION
*Due on Thursday, 17 April.**

Instructions: Give complete answers to receive full credit, including references to any and all sources you used. You may ask the instructor to clarify the instructions or any of the questions, use a calculator or computer to perform any necessary calculations, and consult any sources you wish, *with the exception of other students' work, past or present,* and *you may not give or receive any other aid on this exam, except with the instructor's explicit permission.*

Part I – Various things. Do any *five* (5) of **1** – **9**. *[50 = 5×10 each]*

1. Suppose that A and B are disjoint compact subsets of a complete metric space (X, d). Show that there is a real number $r > 0$ such that for all $a \in A$ and all $b \in B$, $d(a, b) > r$.

2. Suppose that for each $n \geq 0$, $f_n : [0, 1] \to \mathbb{R}$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Does it follow that if $f_n \overset{\text{unif}}{\longrightarrow} f$, then $f : [0, 1] \to \mathbb{R}$ is also differentiable on $(0, 1)$? Prove that it must be or provide (and verify) a counterexample.

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Show that $f^{-1}(A) = \{ x \in \mathbb{R} \mid f(x) \in A \}$ is
- **a.** open if $A \subseteq \mathbb{R}$ is open, *[3]*
- **b.** closed if $A \subseteq \mathbb{R}$ is closed, *[3]*
- **c.** a Borel set if $A \subseteq \mathbb{R}$ is a Borel set. *[4]*

4. Let $c : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ be $c(E) = \begin{cases} \text{the number of elements of } E & \text{if } E \subset \mathbb{R} \text{ is finite} \\ \infty & \text{if } E \subset \mathbb{R} \text{ is infinite} \end{cases}$.
- **a.** Verify that $\mathcal{P}(\mathbb{R}) = \{ E \mid E \subseteq \mathbb{R} \}$ is a σ-algebra on $\mathbb{R}$. *[5]*
- **b.** Verify that c is a measure on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$. *[5]*

5. Suppose $B \subset \mathbb{R}$ is bounded. Show that there is a set $G \subset \mathbb{R}$ such that $B \subseteq G$, G is the intersection of at most countably many open subsets of $\mathbb{R}$, and $|B| = |G|$.

6. Show that the conclusion of Egorov's Theorem can fail if we remove the hypothesis that the common domain of the functions involved is bounded.

7. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a measurable function such that $\displaystyle\int_{\mathbb{R}} |f(x)\| \, dx < \infty$, and define $g : \mathbb{R} \to \mathbb{R}$ by $g(t) = \displaystyle\int_{(-\infty,t)} f(x) \, dx$. Show that g is uniformly continuous on $\mathbb{R}$.

* Please submit your solutions, preferably as a single pdf, via Blackboard's Assignments module. If that fails, please submit them to the instructor on paper or via email to `sbilaniuk@trentu.ca` .

8. Give an example of a continuous function $f : (0,1) \to \mathbb{R}$ such that $\lim_{n\to\infty} \int_{\left(\frac{1}{n},1\right)} f(x)\,dx$ is a real number, but $\int_{(0,1)} f(x)\,dx$ is undefined.

9. Suppose $E \subset \mathbb{R}$ is a measurable set with $|E| < \infty$, $f : E \to [0,\infty)$ is bounded on E, and $\int_E f(x)\,dx = 0$. Show that $|\{\, x \in E \mid f(x) \neq 0 \,\}| = 0$.

[Total = 50]

Part II – Verse thing! Bonus!

ς. Write an original poem touching on metric spaces or measure theory. *[1]*

I hope that you enjoyed the course.
Enjoy the summer!