Mathematics 4790H – Analysis II: Topology and Measure
Trent University, Winter 2025

Assignment #7
How big is the Borel algebra?
Due on Friday, 7 March.[*]

Recall that the cardinality of a set X, which we'll denote by $\|X\|$ to distinguish it from our notation for outer measure, is essentially the number of elements it has, so $\|X\| = \|Y\|$ if and only if all their elements can be paired off. That is, $\|X\| = \|Y\|$ if and only there is a 1–1 onto function $f : X \to Y$. If there is a 1–1 function $g : X \to Y$, then Y has at least as many elements as X does, and we write this as $\|X\| \leq \|Y\|$. Obviously, if $X \subseteq Y$, then $\|X\| \leq \|Y\|$. The Schröder-Bernstein Theorem guarantees that if we have $\|X\| \leq \|Y\|$ and $\|Y\| \leq \|X\|$, then $\|X\| = \|Y\|$. Cantor showed that $\|X\| < \|\mathcal{P}(X)\|$ for any set X, where $\mathcal{P}(X)$ denotes the power set of X.

Recall also that the Borel algebra $\mathcal{B}$ is the smallest σ-algebra on $\mathbb{R}$ which contains all the intervals of $\mathbb{R}$.

1. Show that $\|\mathbb{R}\| \leq \|\mathcal{B}\|$. *[2]*

2. Show that $\|\mathcal{B}\| \leq \|\mathcal{P}(\mathbb{R})\|$. *[1]*

3. $\|\mathcal{B}\|$ is equal to one of $\|\mathbb{R}\|$ or $\|\mathcal{P}(\mathbb{R})\|$. Which is it? Explain why. (You need not give a formal proof.) *[7]*

[*] Please submit your solutions, preferably as a single pdf, via Blackboard's Assignments module. If that fails, please submit them to the instructor on paper or via email to `sbilaniuk@trentu.ca` as soon as you can.