Mathematics 4790H – Analysis II: Topology and Measure
Trent University, Winter 2025

Solutions to Assignment #3
Shrink

Recall that a metric space is said to be *complete* if every Cauchy sequence in the metric space has a limit in the space.

Suppose (X, d) is a metric space. A function $f : X \to X$ is said to be a *contraction mapping* if for some $k \in [0, 1)$ and all x, $y \in X$, $d\left(f(x), f(y)\right) < k \cdot d(x, y)$. *[Note that this forces k to be > 0. If you wish to allow $k = 0$, change the inequality to $d\left(f(x), f(y)\right) \leq k \cdot d(x, y)$.]*

1. Suppose (X, d) is a metric space and $f : X \to X$ is a contraction mapping. Show that f is uniformly continuous on X. *[5]*

Solution. Suppose (X, d) is a metric space and $f : X \to X$ is a contraction mapping with $d\left(f(x), f(y)\right) < k \cdot d(x, y)$ for all x, $y \in X$ and some fixed k with $0 < k < 1$.

Suppose an $\varepsilon > 0$ is given. Let $\delta = \dfrac{\varepsilon}{k}$. Then, for all x, $y \in X$,

$$d(x, y) < \delta = \frac{\varepsilon}{k} \implies d\left(f(x), f(y)\right) < kd(x, y) < k\delta = k \cdot \frac{\varepsilon}{k} = \varepsilon.$$

Since this works for any $\varepsilon > 0$, $f : X \to X$ is uniformly continuous by definition. $\square$

2. Suppose (X, d) is a complete metric space and $f : X \to X$ is a contraction mapping. Show that f has an unique *fixed point*, *i.e.* an $x \in X$ such that $f(x) = x$. *[5]*

Solution. Suppose that (X, d) is a complete metric space and $f : X \to X$ is a contraction mapping with $d\left(f(x), f(y)\right) < k \cdot d(x, y)$ for all x, $y \in X$ and some fixed k with $0 < k < 1$.

We first show that $f : X \to X$ has a fixed point. Choose any $x_0 \in X$ and, given that x_n has been defined, let $x_{n+1} = f\left(x_n\right) = f^{n+1}\left(x_0\right) = f\left(\cdots f\left(x_0\right)\cdots\right)$. Observe first that for any $n \geq 1$,

$$d\left(x_{n+1}, x_n\right) = d\left(f^{n+1}\left(x_0\right), f^n\left(x_0\right)\right) = d\left(f^n\left(f\left(x_0\right)\right), f^n\left(x_0\right)\right) = k^n d\left(x_1, x_0\right).$$

Now suppose that $m > n \geq 1$. Then

$$\begin{aligned}
d\left(x_m, x_n\right) &\leq d\left(x_m, x_{m-1}\right) + d\left(x_{m-1}, x_{m-2}\right) + \cdots + d\left(x_{n+1}, x_n\right) \\
&< k^{m-1} d\left(x_1, x_0\right) + k^{m-2} d\left(x_1, x_0\right) + \cdots + k^n d\left(x_1, x_0\right) \\
&= \left(k^{m-1} + k^{m-2} + \cdots + k^n\right) d\left(x_1, x_0\right) \\
&< k^n \left(k^{m-n-1} + k^{m-n-2} + \cdots + k^1 + k^0\right) d\left(x_1, x_0\right) \\
&= k^n \left(k^{m-n-1} + k^{m-n-2} + \cdots + k + 1\right) d\left(x_1, x_0\right) \\
&= k^n \left(\sum_{i=0}^{m-n-1} k^i\right) d\left(x_1, x_0\right) \\
&< k^n \left(\sum_{i=0}^{\infty} k^i\right) d\left(x_1, x_0\right) = k^n \cdot \frac{1}{1-k} \cdot d\left(x_1, x_0\right) = \frac{k^n d\left(x_1, x_0\right)}{1-k}.
\end{aligned}$$

It follows that $\{x_n\}$ is a Cauchy sequence. Since $0 < k < 1$, $k^n \to 0$ as $n \to \infty$, so if $\varepsilon > 0$, we need only choose an $N \geq 1$ large enough to get $k^N < \dfrac{1-k}{d(x_1, x_0)}\varepsilon$ to ensure that if $m > n \geq N$, we have

$$d(x_m, x_n) < \frac{k^n d(x_1, x_0)}{1-k} \leq \frac{k^N d(x_1, x_0)}{1-k} < \frac{1-k}{d(x_1, x_0)}\varepsilon \cdot \frac{d(x_1, x_0)}{1-k} = \varepsilon.$$

Since (X, d) is a complete metric space, it follows that the sequence $\{x_n\}$ has a limit, say x^*. We claim that x^* is a fixed point of the contraction mapping $f : X \to X$. By the result in question **1**, f must be continuous, so

$$x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f\left(\lim_{n\to\infty} x_{n-1}\right) = f(x^*),$$

i.e. x^* is a fixed point of $f : X \to X$.

It remains to show that x^* is the only fixed point of $f : X \to X$. Suppose, by way of contradiction, x^* and y^* are different fixed points of the contraction mapping f. Then we would have

$$d(x^*, y^*) = d(f(x^*), f(y^*)) < kd(x^*, y^*) < d(x^*, y^*).$$

This is a contradiction because no real number can be structly less than itself. Thus x^* must be the unique fixed point of $f : X \to X$. ■