Mathematics 4790H – Analysis II: Topology and Measure
Trent University, Winter 2025

Solutions to Assignment #~~1~~ 2
Cases of Completeness

Recall that a metric space is said to be *complete* if every Cauchy sequence in the metric space has a limit in the space.

$L^\infty([0,1])$ is the metric space whose points are all the continuous functions $f:[0,1]\to\mathbb{R}$, equipped with the metric $d(f,g)=\sup\{\,|f(x)-g(x)|\,0\le x\le 1\,\}$, sometimes written as $d(f,g)=\max\{\,|f(x)-g(x)|\,0\le x\le 1\,\}$. On the other hand, $L^1([0,1])$ is the metric space which has the same points, *i.e.* the continuous functions $f:[0,1]\to\mathbb{R}$, but equipped with the metric $d(f,g)=\displaystyle\int_0^1 |f(x)-g(x)|\,dx$

1. Show that $L^\infty([0,1])$ is complete. *[5]*

Solution. Suppose $\{f_n\}$ is a Cauchy sequence in $L^\infty([0,1])$.

We claim first that it converges pointwise to some function $f:[0,1]\to\mathbb{R}$. Suppose $t\in[0,1]$ and some $\varepsilon>0$ is given. Since $\{f_n\}$ is Cauchy, there is an N such that if $m,n\ge N$, then $d(f_n,f_m)<\varepsilon$. But then , for m, $n\ge N$, we have $|f_n(t)-f_m(t)|\le\sup\{\,|f_n(x)-f_m(x)|\,0\le x\le 1\,\}=d(f_n,f_m)$. Thus, for any $t\in[0,1]$, the sequence $\{f_n(t)\}$ is Cauchy in $\mathbb{R}$ (with the usual metric). As $\mathbb{R}$ is a complete metric space, it follows that for each $t\in[0,1]$, $f_n(t)\to f(t)$ for some $f(t)\in\mathbb{R}$, as required for pointwise convergence.

In fact, $f_n\to f$ uniformly. Suppose $\varepsilon>0$ is given. Choose an N large enough so that if m, $n\ge N$, then $d(f_n,f_m)<\frac{\varepsilon}{2}$. Then for all $t\in[0,1]$ at once we have

$$|f_n(t)-f_m(t)|\le\sup\{\,|f_n(x)-f_m(x)|\,0\le x\le 1\,\}=d(f_n,f_m)<\frac{\varepsilon}{2}.$$

In particular, this will be true for any particular $n\ge N$ and all $t\in[0,1]$ as $m\to\infty$ and $f_m(t)\to f(t)$, so it follows that $|f_n(t)-f(t)|\le\frac{\varepsilon}{2}<\varepsilon$. Since this is true for all $n\ge N$ and all $t\in[0,1]$, $f_n\to f$ uniformly on $[0,1]$.

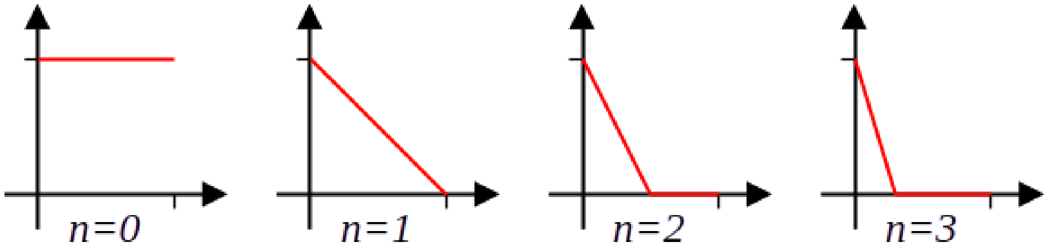
Since the uniform limit of continuous functions is continuous, it follows from the above that $f:[0,1]\to\mathbb{R}$ is continuous [see *e.g.* Theorem 5.6 in the textbook *An Introduction to Metric Spaces*], *i.e.* $f\in C([0,1])$ is a point of $L^\infty([0,1])$. Thus the Cauchy sequence $\{f_n\}$ has a limit in $L^\infty([0,1])$. Since this can be done for any Cauchy sequence in the space, $L^\infty([0,1])$is complete. □

2. Show that $L^1([0,1])$ is not complete. *[5]*

Solution. Consider the sequence of functions $\{f_n\}$, where $f_0:[0,1]\to\mathbb{R}$ is given by $f_0(x)=1$ for all $x\in[0,1]$, and $f_n:[0,1]\to\mathbb{R}$ for $n\ge 1$ is given by

$$f_n(x)=\begin{cases}1-nx & 0\le x\le\frac{1}{n}\\ 0 & \frac{1}{n}\le x\le 1\end{cases}.$$

The graphs of the first four functions in the sequence look like this:



Since $1 - n\frac{1}{n} = 1 - 1 = 0$ for all $n > 0$, $f_n(x)$ is indeed well-defined and continuous at $x = \frac{1}{n}$; since it is otherwise pieced together from two linear functions, it is continuous for all $x \in [0, 1]$.

It is easy to see that $f_n(0) = 1$ for all x and that $f_n(x) = 0$ for $x \geq \frac{1}{n}$. It follows that $\lim_{n\to\infty} f_n(0) = 1$ and that $\lim_{n\to\infty} f_n(x) = 0$ for all x such that $0 < x \leq 1$. Thus the pointwise limit of the sequence $\{f_n\}$ is the function $f : [0, 1] \to \mathbb{R}$ given by $f(x) = \begin{cases} 1 & x = 0 \\ 0 & 0 < x \leq 1 \end{cases}$, which is obviously discontinuous at $x = 0$ and hence is *not* a point of $L^1([0, 1])$. This means that the sequence $\{f_n\}$ has no limit in $L^1([0, 1])$.

On the other hand, $\{f_n\}$ is a Cauchy sequence in $L^1([0, 1])$. It is not hard to see – compare the graphs in the diagram above – that if $m > n \geq 1$, then

$$d(f_n, f_m) = \int_0^1 |f_n(x) - f(_m(x)|\, dx < \int_0^1 f_n(x)\, dx = \frac{1}{2} \cdot \frac{1}{n} \cdot 1 = \frac{1}{2n}.$$

It follows quickly that for any $\varepsilon > 0$ there is an N – any $N > \dfrac{1}{2\varepsilon}$ will do – such that if $m > n \geq N$, then $d(f_n, f_m) < \dfrac{1}{2n} \leq \dfrac{1}{2N} < \varepsilon$. Thus $\{f_n\}$ is indeed a Cauchy sequence in $L^1([0, 1])$.

Since not every Cauchy sequence in $L^1([0, 1])$ converges, it is not complete. ■