Mathematics 4790H – Analysis II: Topology and Measure
Trent University, Winter 2025

Solutions to Assignment #1
Playing With Sequences

Suppose $\left[a_n^k\right]$ is an infinite matrix of real numbers such that for all $n \geq 0$, $\lim_{k\to\infty} a_n^k = a_n$ for some real number a_n, and for all $k \geq 0$, $\lim_{n\to\infty} a_n^k = a^k$ for some real number a^k.

$$\begin{array}{cccccccc}
a_0^0 & a_1^0 & a_2^0 & a_3^0 & a_4^0 & \cdots & \to & a^0 \\
a_0^1 & a_1^1 & a_2^1 & a_3^1 & a_4^1 & \cdots & \to & a^1 \\
a_0^2 & a_1^2 & a_2^2 & a_3^2 & a_4^2 & \cdots & \to & a^2 \\
a_0^3 & a_1^3 & a_2^3 & a_3^3 & a_4^3 & \cdots & \to & a^3 \\
a_0^4 & a_1^4 & a_2^4 & a_3^4 & a_4^4 & \cdots & \to & a^4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & & \downarrow \\
a_0 & a_1 & a_2 & a_3 & a_4 & \cdots & \to & ?
\end{array}$$

1. Give an example to show that even if $\lim_{n\to\infty} a_n$ and $\lim_{k\to\infty} a^k$ both exist, they need not be equal. *[5]*

Solution. Consider the infinte matrix given by $a_n^k = \begin{cases} 1 & n < k \\ 0 & n \geq k \end{cases}$ for n, $k \in \mathbb{N}$, *i.e.*

$$\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & \cdots & \to & 0 \\
1 & 0 & 0 & 0 & 0 & \cdots & \to & 0 \\
1 & 1 & 0 & 0 & 0 & \cdots & \to & 0 \\
1 & 1 & 1 & 0 & 0 & \cdots & \to & 0 \\
1 & 1 & 1 & 1 & 0 & \cdots & \to & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & & \downarrow \\
1 & 1 & 1 & 1 & 1 & \cdots & \to & 1 \neq 0
\end{array}$$

It is pretty clear that $a^k = \lim_{n\to\infty} a_n^k = \lim_{n\geq k} 0 = 0$ for all k, and that $a_n = \lim_{k\to\infty} a_n^k = \lim_{k>n} 1 = 1$ for all n. Thus $\lim_{n\to\infty} a_n = 1$ and $\lim_{k\to\infty} a^k = 0$ both exist, but are not equal. $\square$

2. Give a condition on the limits involved in this setup that ensures that $\lim_{n\to\infty} a_n = \lim_{k\to\infty} a^k$ if both limits exist, and prove that it does. *[5]*

SOLUTION. Having

- the sequences $\left\{a_n^k \mid n \geq 0\right\}$ converge uniformly to the sequence $\{a_n \mid n \geq 0\}$ as $k \to \infty$, and
- the sequences $\left\{a_n^k \mid k \geq 0\right\}$ converge uniformly to the sequence $\left\{a^k \mid k \geq 0\right\}$ as $n \to \infty$,

will ensure that $\lim_{n\to\infty} a_n = \lim_{k\to\infty} a^k$ if both of these limits exist.

The sequences $\left\{a_n^k \mid n \geq 0\right\}$ converge uniformly to $\{a_n \mid n \geq 0\}$ as $k \to \infty$ means that for all $\varepsilon > 0$, there is a K such that for all $k \geq K$, $\left|a_n^k - a_n\right| < \varepsilon$ for all $n \geq 0$ simultaneously. Similarly, the sequences $\left\{a_n^k \mid k \geq 0\right\}$ converge uniformly to $\left\{a^k \mid k \geq 0\right\}$ as $n \to \infty$ means that for all $\varepsilon > 0$, there is an N such that for all $n \geq N$, $\left|a_n^k - a_n\right| < \varepsilon$ for all $k \geq 0$ simultaneously.

We check that this condition does do the job; that is, if the rows and columns of the matrix converge uniformly to the sequences $\{a_n\}$ and $\left\{a^k\right\}$, respectively, then $\lim_{n\to\infty} a_n = \lim_{k\to\infty} a^k$ if both of these limits exist. Suppose then that we have the uniform convergence, and that $\lim_{n\to\infty} a_n = b$ and $\lim_{k\to\infty} a^k = c$. We need to show that $b = c$, which we will do by showing that $|b - c| < \varepsilon$ for any $\varepsilon > 0$. Suppose that an $\varepsilon > 0$ is given.

Since the sequences $\left\{a_n^k \mid n \geq 0\right\}$ converge uniformly to $\{a_n \mid n \geq 0\}$ as $k \to \infty$, there is a K such that for all $k \geq K$, $\left|a_n^k - a_n\right| < \frac{\varepsilon}{4}$ for all $n \geq 0$ simultaneously.

Since the sequences $\left\{a_n^k \mid k \geq 0\right\}$ converge uniformly to $\left\{a^k \mid k \geq 0\right\}$ as $n \to \infty$, there is an N such that for all $n \geq N$, $\left|a_n^k - a_n\right| < \frac{\varepsilon}{4}$ for all $k \geq 0$ simultaneously

Since $\lim_{n\to\infty} a_n = b$, there is an S such that for all $n \geq S$, $|a_n - b| < \frac{\varepsilon}{4}$.

Since $\lim_{k\to\infty} a^k = c$, there is a T such that for all $k \geq T$, $\left|a^k - c\right| < \frac{\varepsilon}{4}$.

Let $M = \max\{K, N, S, T\}$. Then if we have $k, n \geq M$, it follows that each of $\left|a_n^k - a_n\right|$, $\left|a_n^k - a_n\right|$, $|a_n - b|$, and $\left|a^k - c\right|$ will be less than $\frac{\varepsilon}{4}$. With the help of the triangle inequality, it then follows that:

$$\begin{aligned}
|b - c| &< \left|b - a_n + a_n - a^k + a^k - c\right| \\
&\leq |b - a_n| + \left|a_n - a^k\right| + \left|a^k - c\right| \\
&= |a_n - b| + \left|a_n - a_n^k + a_n^k - a^k\right| + \left|a^k - c\right| \\
&\leq |a_n - b| + \left|a_n - a_n^k\right| + \left|a_n^k - a^k\right| + \left|a^k - c\right| \\
&\leq |a_n - b| + \left|a_n^k - a_n\right| + \left|a_n^k - a^k\right| + \left|a^k - c\right| \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon
\end{aligned}$$

Since $|b - c| < \varepsilon$ for any $\varepsilon > 0$, we must have $0 \leq |b - c| \leq 0$, which is only possible if $b = c$. Hence $\lim_{n\to\infty} a_n = b = c = \lim_{k\to\infty} a^k$, as required. ■