Mathematics-Computer Science 4215H – Mathematical Logic
TRENT UNIVERSITY, Winter 2021

Assignment #5
Due on Friday, 26 February.

Do all of the following problems, two of which are straight out of the textbook[0] (which explains the numbering), reproduced here for your convenience.

4.12. *[Theorem 4.12 – Completeness Theorem]* If Δ is a set of formulas and α is a formula such that $\Delta \vDash \alpha$, then $\Delta \vdash \alpha$. *[5]*

SOLUTION. It suffices to prove the contrapositive, *i.e.* that if $\Delta \nvdash \alpha$, then $\Delta \nvDash \alpha$.

Suppose that $\Delta \nvdash \alpha$. We claim that $\Delta \cup \{\neg\alpha\}$ must be consistent. Assume, by way of contradiction, that $\Delta \cup \{\neg\alpha\}$ is inconsistent, *i.e.* $\Delta \cup \{\neg\alpha\} \vdash \neg(\varphi \to \varphi)$ for some formula φ. It follows by the Deduction Theorem that $\Delta \vdash \neg\alpha \to \neg(\varphi \to \varphi)$, say via the deduction $\eta_1\eta_2\dots\eta_n$. By Problem 3.9(3), $\vdash (\neg\alpha \to \neg(\varphi \to \varphi)) \to ((\varphi \to \varphi) \to \alpha)$, say via the deduction $\xi_1\xi_2\dots\xi_k$. By Example 3.1, $\vdash \varphi \to \varphi$, via a deduction of length 5 given in the example which we shall abbreviate as $\tau_1 \dots \tau_5$ here. Then

$$\eta_1\eta_2\dots\eta_n\xi_1\xi_2\dots\xi_k\,((\varphi \to \varphi) \to \alpha)\,\tau_1\dots\tau_5\alpha$$

is a deduction of α from Δ. Note that $((\varphi \to \varphi) \to \alpha)$ follows from η_n and ξ_k by Modus Ponens, and α follows from $((\varphi \to \varphi) \to \alpha)$ and τ_5 by Modus Ponens. Thus if $\Delta \cup \{\neg\alpha\}$ is inconsistent, $\Delta \vdash \alpha$ is true, but this contradicts $\Delta \nvdash \alpha$.

Hence if $\Delta \nvdash \alpha$, then $\Delta \cup \{\neg\alpha\}$ must be consistent. By Theorem 4.11, it follows that $\Delta \cup \{\neg\alpha\}$ is satisfiable, but any truth assignment satisfying this set makes every formula in Δ true and makes α false, so $\Delta \nvDash \alpha$, as desired. ■

4.13. *[Theorem 4.13 – Compactness Theorem]* A set of formulas Γ is satisfiable if and only if every finite subset of Γ is satisfiable. *[4]*

SOLUTION. Suppose Γ is a set of formulas of $\mathcal{L}_P$. Then

$$\begin{aligned}\Gamma \text{ is satisfiable} &\iff \Gamma \text{ is consistent [Theorem 4.11]}\\ &\iff \text{Every finite subset } \Delta \text{ of } \Gamma \text{ is consistent [Corollary 4.5]}\\ &\iff \text{Every finite subset } \Delta \text{ of } \Gamma \text{ is satisfiable [Theorem 4.11]} \quad ■\end{aligned}$$

In proving the above results, you may appeal to any preceding results and problems in the textbook that you like. Given that, both should be fairly easy to put away. The following application of the Compactness Theorem (Hint!), will require probably greater effort.

[0] *A Problem Course in Mathematical Logic*, Version 1.6.

RT. *[Ramsey's Theorem]* For every integer $n > 0$ there is an integer $R_n > 0$ such that if $G = (V, E)$ is a graph with at least R_n vertices, then G has a clique of size n or an independent set of size n. *[6]*

This problem requires some background, some of which you have probably seen elsewhere:

- A *graph* G is a pair (V, E) consisting of a set V of *vertices* and set $E \subset V \times V$ of *edges* such that $(u, v) \in E \iff (v, u) \in E$.
- A *clique* of a graph $G = (V, E)$ is a subset $C \subseteq V$ of the vertices such that for all $u,\ v \in C$, $(u, v) \in E$.
- An *independent set* of a $G = (V, E)$ is a subset $I \subseteq V$ of the vertices such that for all $u,\ v \in I$, $(u, v) \notin E$.
- *[Infinite Ramsey's Theorem]* A graph $G = (V, E)$ with infinitely many vertices must have an infinite clique or an infinite independent set. [You may, and will probably need to, assume this theorem.]

Your task over Reading Week is to try to figure out how the Compactness Theorem could be useful in proving Ramsey's Theorem. I will be forthcoming with hints *after* Reading Week ...

SOLUTION. Our strategy will be show that the failure of Ramsey's Theorem for some n leads to a contradiction. The Compactness Theorem will allow us to proceed from a failure of Ramsey's Theorem for finite graphs to an infinite graph contradicting the Infinite Ramsey's Theorem. What we need to make this happen is a way of representing graphs using propositional logic.

To deal with arbitrarily large finite – and possibly (countably) infinite – graphs we need a set of countably infinitely many vertices, say $V = \{ v_0, v_1, v_2, \dots \} = \{ v_i \mid i \in \mathbb{N} \}$. A set $E \subseteq V \times V$ of edges for a graph using some or all of these vertices can then be represented by atomic formulas as follows.

Let $\sigma : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a 1–1 onto function.[1] We will use $A_{\sigma(i,j)}$ to represent an edge between the vertices v_i and v_j: if $A_{\sigma(i,j)}$ is true, then $(v_i, v_j) \in E$, and if $A_{\sigma(i,j)}$ is false, then $(v_i, v_j) \notin E$. Thus each possible graph $G(V, E)$ on our set of vertices corresponds to a truth assignment in our propositional logic.

Suppose now that Ramsey's Theorem fails for some $n > 0$. That is, suppose that there are arbitrarily large finite graphs which have neither a clique of size n nor an independent set of size n. Let Σ_0, Σ_1, and Σ_2 be the following sets of formulas of $\mathcal{L}_P$:

$$\Sigma_0 = \{ A_{\sigma(i,j)} \leftrightarrow A_{\sigma(j,i)} \mid i,\ j \in \mathbb{N} \}$$
$$\Sigma_1 = \{ \neg \left(A_{\sigma(i_1,i_2)} \land A_{\sigma(i_1,i_3)} \land \cdots \land A_{\sigma(i_{n-1},i_n)} \right) \mid i_1,\ i_2, \dots,\ i_n \in \mathbb{N} \ \&\ \text{are distinct} \}$$
$$\Sigma_2 = \{ A_{\sigma(i_1,i_2)} \lor A_{\sigma(i_1,i_3)} \lor \cdots \lor A_{\sigma(i_{n-1},i_n)} \mid i_1,\ i_2, \dots,\ i_n \in \mathbb{N} \ \&\ \text{are distinct} \}$$

The formulas in Σ_0 assure us that we are dealing with a graph by ensuring that we satisfy the condition that edges be bi-directional, *i.e.* $(u, v) \in E \iff (v, u) \in E$. Σ_1 consists of formulas that between them assert that every collection of n vertices v_{i_1}, v_{i_2}, ... , v_{i_n},

[1] If you need to be totally explicit for some reason, $\sigma(i, j) = (i + j)(i + j + 1)/2 + j$ will do.

has at least one missing edge, so it is not a clique. Similarly, Σ_2 consists of formulas that between them assert that every collection of n vertices v_{i_1}, v_{i_2}, $\ldots$, v_{i_n}, has at least one edge, so it is not an independent set.

Let $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$. We claim that if Ramsey's Theorem fails for n, then Σ is satisfiable. Suppose Δ is a finite subset of Σ. Since Δ is finite, there is a largest integer $m \geq 0$ such that v_m occurs in one of the finitely many possible edges corresponding to one of the atomic formulas occurring in Δ. Since Ramsey's Theorem supposedly fails for n, there is a graph $G' = (V', E')$ with $m+1$ (or more vertices) that has no clique of size n and no independent set of size n. Identify $m+1$ vertices of G' with v_1, v_2, $\ldots$, and v_m, respectively, and define a truth assignment u by $u\left(A_\sigma(i,j)\right) = T$, for i, $j \leq m$, if and only if there is an edge between the vertices of G' corresponding to v_i and v_j. Define u in any way you like for all other atomic formulas. It is not hard to see that u must satisfy Δ: Every formula in Δ of the form ...

- $A_{\sigma(i,j)} \leftrightarrow A_{\sigma(j,i)}$ is satisfied because G' is a graph, so $(u,w) \in E' \iff (w,u) \in E'$;
- $\neg\left(A_{\sigma(i_1,i_2)} \wedge A_{\sigma(i_1,i_3)} \wedge \cdots \wedge A_{\sigma(i_{n-1},i_n)}\right)$ is satisfied because G' has no cliques of size n, so every collection of n of it's vertices is missing at least one edge; and
- $A_{\sigma(i_1,i_2)} \vee A_{\sigma(i_1,i_3)} \vee \cdots \vee A_{\sigma(i_{n-1},i_n)}$ is satisfied because G' has no independent sets of size n, so every collection of n of it's vertices has at least one edge.

Since every finite subset of Σ is satisfiable, it follows by the Compactness Theorem that Σ is satisfiable.

However, if Σ is satisfied by some truth assignment, this corresponds to an infinite graph $G = (V, E)$ that has no cliques of size n and no independent sets of size n. This contradicts the Infinite Ramsey's Theorem, since an infinite graph must have an infinite clique or an infinite independent set: the former would give you cliques of every finite size, including n, and the latter would give you independent sets of every finite size, including n.

Since having Σ be satisfiable leads to a contradiction and having Ramsey's Theorem fail for n leads to having Σ be satisfiable, Ramsey's Theorem must be true for n. This reasoning for every $n > 0$, so Ramsey's Theorem must be true. ■

[Total = 15]