Mathematics-Computer Science 4215H – Mathematical Logic
TRENT UNIVERSITY, Winter 2021

Solutions to Assignment #4
Due on Friday, 12 February.

Do all of the following problems, which are straight out of the textbook[0] (which explains the numbering), reproduced here for your convenience.

3.9. *[Problem 3.9]* Appealing to previous deductions and the Deduction Theorem if you wish, show that:

(1) $\{\delta, \neg\delta\} \vdash \gamma$ *[1]*
(2) $\vdash \varphi \to \neg\neg\varphi$ *[4]*

NOTE. You may assume any and all of the examples, problems, and results of Chapter 3, up to 3.8 inclusive, when doing **3.9** (1) & (2).

SOLUTIONS. 3.9(1): We will show that $\{\delta, \neg\delta\} \vdash \gamma$ via the following deduction:

1.	$(\neg\gamma \to \neg\delta) \to ((\neg\gamma \to \delta) \to \gamma)$	(A3)
2.	$\neg\delta \to (\neg\gamma \to \neg\delta)$	(A1)
3.	$\neg\delta$	Premiss
4.	$\neg\gamma \to \neg\delta$	2, 3 MP
5.	$(\neg\gamma \to \delta) \to \gamma$	1, 4 MP
6.	$\delta \to (\neg\gamma \to \delta)$	(A1)
7.	δ	Premiss
8.	$\neg\gamma \to \delta$	6, 7 MP
9.	γ	5, 8 MP

One down! $\square$

3.9(2): By the Deduction Theorem, to show $\vdash \varphi \to \neg\neg\varphi$ it is sufficient to show that $\{\varphi\} \vdash \neg\neg\varphi$. We show the latter via the following deduction:

1.	$(\neg\neg\neg\varphi \to \neg\varphi) \to ((\neg\neg\neg\varphi \to \varphi) \to \neg\neg\varphi)$	(A3)
2.	$\neg\neg\neg\varphi \to \neg\varphi$	Example 3.4
3.	$(\neg\neg\neg\varphi \to \varphi) \to \neg\neg\varphi$	1, 2 MP
4.	$\varphi \to (\neg\neg\neg\varphi \to \varphi)$	(A1)
5.	φ	Premiss
6.	$\neg\neg\neg\varphi \to \varphi$	4, 5 MP
7.	$\neg\varphi$	3, 6 MP

Two down! $\blacksquare$

4.3. *[Proposition 4.3]* Suppose Δ is an inconsistent set of formulas. Then $\Delta \vdash \psi$ for any formula ψ. *[2]*

SOLUTION. Suppose Δ is an inconsistent set of formulas and ψ is any formula. By the definition of "inconsistent", $\Delta \vdash \neg(\alpha \to \alpha)$ for some formula α, say via the deduction $\eta_1\eta_2\ldots\eta_n$. Then

[0] *A Problem Course in Mathematical Logic*, Version 1.6.

$$
\begin{array}{rll}
1. & \eta_1 & \\
\vdots & \vdots & \\
n. & \eta_n \quad [\textit{i.e. } \neg(\alpha \to \alpha)] & \\
n+1. & (\neg\psi \to \neg(\alpha \to \alpha)) \to ((\neg\psi \to (\alpha \to \alpha)) \to \psi) & \text{(A3)} \\
n+2. & \neg(\alpha \to \alpha) \to (\neg\psi \to \neg(\alpha \to \alpha)) & \text{(A1)} \\
n+3. & \neg\psi \to \neg(\alpha \to \alpha) & n,\ n+2 \text{ MP} \\
n+4. & (\neg\psi \to (\alpha \to \alpha)) \to \psi & n+1,\ n+3 \text{ MP} \\
n+5. & (\alpha \to \alpha) \to (\neg\psi \to (\alpha \to \alpha)) & \text{(A1)} \\
n+6. & \alpha \to \alpha & \text{Example 3.1} \\
n+7. & \neg\psi \to (\alpha \to \alpha) & n+5,\ n+6 \text{ MP} \\
n+8. & \psi & n+4,\ n+7 \text{ MP}
\end{array}
$$

is a deduction of ψ from Δ, as required. ■

4.4. *[Proposition 4.4]* Suppose Σ is an inconsistent set of formulas. Then there is a finite subset Δ of Σ such that Δ is inconsistent. *[2]*

Solution. Suppose Σ is an inconsistent set of formulas. By definition, this means that $\Sigma \vdash \neg(\alpha \to \alpha)$ for some formula α, say via the deduction $\eta_1\eta_2\dots\eta_n$. Let $\Delta = \{\, \eta_i \mid \eta_i \in \Sigma \,\}$. $\Delta \subseteq \Sigma$ by its definition and Δ is finite because it is also a subset of the finite set $\{\, \eta_1, \eta_2, \dots, \eta_n \,\}$. Finally, $\Delta \vdash \neg(\alpha \to \alpha)$ because the deduction $\eta_1\eta_2\dots\eta_n$ is a deduction of $\neg(\alpha \to \alpha)$ as well: every time some η_i is justified as being a premiss from Σ, it is also present in Δ. Thus Δ is a finite inconsistent subset of Σ. ■

4.5. *[Corollary 4.5]* A set of formulas Γ is consistent if and only if every finite subset of Γ is consistent. *[2]*

Solution. ($\Longleftarrow$) Suppose every finite subset of Γ is consistent. Then Γ must be consistent because if it was inconsistent, some finite subset of it would also be inconsistent by Proposition 4.4.

($\Longrightarrow$) Suppose Γ is consistent. If some finite subset $\Delta \subseteq \Gamma$ were inconsistent, then, be definition, $\Delta \vdash \neg(\alpha \to \alpha)$ for some formula α. By Proposition 3.6, it would follow that $\Gamma \vdash \neg(\alpha \to \alpha)$ because $\Delta \subseteq \Gamma$, contradicting the fact that Γ is consistent. Thus every finite subset of Γ must also be consistent. ■

4.8. *[Proposition 4.8]* Suppose Σ is a maximally consistent set of formulas and φ is a formula. Then $\neg\varphi \in \Sigma$ if and only if $\varphi \notin \Sigma$. *[2]*

Solution. ($\Longrightarrow$) Suppose Σ is a maximally consistent set of formulas and $\neg\varphi \in \Sigma$. By Problem 3.9(1), $\{\varphi, \neg\varphi\} \vdash \neg(\alpha \to \alpha)$ for any formula α, so $\{\varphi, \neg\varphi\}$ is inconsistent. If we had $\varphi \in \Sigma$, then would be an inconsistent finite subset of Σ, which would make Σ inconsistent by Corollary 4.5. However, this would contradict the maximal consistency of Σ, so we must have $\varphi \notin \Sigma$.

($\Longleftarrow$) Suppose Σ is a maximally consistent set of formulas and $\varphi \notin \Sigma$. By the definition of maximal consistency, it follows that $\Sigma \cup \{\varphi\}$ is inconsistent, and so, by Proposition 4.3,

$\Sigma \cup \{\varphi\} \vdash \neg(\alpha \to \alpha)$ for some (indeed, any) formula α. Applying the Deduction Theorem, it now follows that $\Sigma \vdash \varphi \to \neg(\alpha \to \alpha)$, say via the deduction $\mu_1\mu_2 \ldots \mu_n$. Then

1.	μ_1	
$\vdots$	$\vdots$	
n.	μ_n [*i.e.* $\varphi \to \neg(\alpha \to \alpha)$]	
$n+1$.	$(\varphi \to \neg(\alpha \to \alpha)) \to ((\alpha \to \alpha) \to \neg\varphi)$	Problem 3.9(6)
$n+2$.	$(\alpha \to \alpha) \to \neg\varphi$	n, $n+1$ MP
$n+3$.	$\alpha \to \alpha$	Example 3.1
$n+4$.	$\neg\varphi$	$n+2$, $n+3$ MP

is a deduction of $\neg\varphi$ from Σ, so $\Sigma \vdash \neg\varphi$. Since Σ is maximally consistent it follows by Proposition 4.7 that $\neg\varphi \in \Sigma$. ■

4.9. *[Proposition 4.9]* Suppose Σ is a maximally consistent set of formulas and φ and ψ are formulas. Then $\varphi \to \psi \in \Sigma$ if and only if $\varphi \notin \Sigma$ or $\psi \in \Sigma$. *[2]*

Solution. ($\Longrightarrow$) Suppose Σ is a maximally consistent set of formulas and $\varphi \to \psi \in \Sigma$. We need to show that $\varphi \notin \Sigma$ or $\psi \in \Sigma$; it suffices to show that if $\varphi \notin \Sigma$ fails, *i.e.* $\varphi \in \Sigma$, then $\psi \in \Sigma$. Assuming $\varphi \in \Sigma$,

1.	$\varphi \to \psi$	Premiss
2.	φ	Premiss
3.	ψ	1, 2 MP

is a deduction of ψ from Σ, so $\Sigma \vdash \psi$. Since Σ is maximally consistent it follows by Proposition 4.7 that $\psi \in \Sigma$.

($\Longleftarrow$) Suppose Σ is a maximally consistent set of formulas, with $\varphi \notin \Sigma$ or $\psi \in \Sigma$.

First, if $\psi \in \Sigma$, then

1.	$\psi \to (\varphi \to \psi)$	(A1)
2.	ψ	Premiss
3.	$\varphi \to \psi$	1, 2 MP

is a deduction of $\varphi \to \psi$ from Σ. Since Σ is maximally consistent it follows by Proposition 4.7 that $\varphi \to \psi \in \Sigma$.

Second, if $\varphi \notin \Sigma$, then $\neg\varphi \in \Sigma$ by Proposition 4.8. Then

1.	$\neg\varphi \to (\neg\psi \to \neg\varphi)$	(A1)
2.	$\neg\varphi$	Premiss
3.	$\neg\psi \to \neg\varphi$	1, 2 MP
4.	$(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$	Problem 3.9(3)
5.	$\varphi \to \psi$	3, 4 MP

is a deduction of $\varphi \to \psi$ from Σ. Since Σ is maximally consistent it follows by Proposition 4.7 that $\varphi \to \psi \in \Sigma$.

Either way, if $\varphi \notin \Sigma$ or $\psi \in \Sigma$, then $\varphi \to \psi \in \Sigma$, as required. ■

[Total = 15]