Mathematics-Computer Science 4215H – Mathematical Logic
TRENT UNIVERSITY, Winter 2021

Assignment #3
Due on Friday, 5 February.

Do all of the following problems, which are straight out of the textbook[0] (which explains the numbering), reproduced here for your convenience.

3.1. *[Proposition 3.1]* Every axiom of $\mathcal{L}_P$ is a tautology. *[3]*

SOLUTION. We will use truth tables to check:

(A1)

α	β	$(\alpha \to (\beta \to \alpha))$	
T	T	**T**	T
T	F	**T**	T
F	T	**T**	F
F	F	**T**	T

(A2)

α	β	γ	$((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$					
T	T	T	T	T	**T**	T	T	T
T	T	F	F	F	**T**	T	F	F
T	F	T	T	T	**T**	F	T	T
T	F	F	T	T	**T**	F	T	F
F	T	T	T	T	**T**	T	T	T
F	T	F	T	F	**T**	T	T	T
F	F	T	T	T	**T**	T	T	T
F	F	F	T	T	**T**	T	T	T

(A3)

α	β	$(((\neg\beta) \to (\neg\alpha)) \to (((\neg\beta) \to \alpha) \to \beta))$						
T	T	F	T	F	**T**	F	T	T
T	F	T	F	F	**T**	T	T	F
F	T	F	F	T	**T**	F	T	T
F	F	F	T	T	**T**	T	F	T

Since no matter what truth values a truth assignment may give the subformulas in each of the axiom schema, it makes that axiom true, every axiom of $\mathcal{L}_P$ is a tautology. ■

3.2. *[Proposition 3.2]* Suppose φ and ψ are formulas. Then $\{\, \varphi, (\varphi \to \psi) \,\} \vdash \psi$. *[1]*

SOLUTION. There was a typo in the version of the proposition given on the assignment. Proposition 3.2, as correctly given in the textbook, has $\{\, \varphi, (\varphi \to \psi) \,\} \vDash \psi$ as the conclusion instead of $\{\, \varphi, (\varphi \to \psi) \,\} \vdash \psi$. Given that, I accepted solutions to both versions. Solutions to both versions are given below.

[0] *A Problem Course in Mathematical Logic*, Version 1.6.

($\vdash$) $\{\, \varphi,\, (\varphi \to \psi)\,\} \vdash \psi$ via the following deduction:

1.	φ	Premiss
2.	$\varphi \to \psi$	Premiss
3.	ψ	1,2 MP

That's that! □

($\vDash$) We check that $\{\, \varphi,\, (\varphi \to \psi)\,\} \vDash \psi$ using a truth table:

φ	ψ	$\varphi \to \psi$
T	T	T
T	F	F
F	T	T
F	F	T

In every row (that is, in the first row) of the truth table that both φ and $\varphi \to \psi$ are true, we also have ψ true, every truth assignment satisfying $\{\, \varphi,\, (\varphi \to \psi)\,\}$ also satisfies ψ, as required. ■

3.4. *[Proposition 3.4]* If $\varphi_1\varphi_2\ldots\varphi_n$ is a deduction of $\mathcal{L}_P$, then $\varphi_1\ldots\varphi_\ell$ is also a deduction of $\mathcal{L}_P$ for any ℓ such that $1 \leq \ell \leq n$. *[1]*

Solution. If $\varphi_1\varphi_2\ldots\varphi_n$ is a deduction from set of formulas Σ, then, by definition, for each $k \leq n$, φ_k is a premiss (*i.e.* in Σ), an axiom, or there are $i, j < k$ such that φ_k follows from φ_i and φ_j by MP. Since $\ell \leq n$, it follows that for each $k \leq \ell$, φ_k is a premiss from Σ, an axiom, or there are $i, j < k$ such that φ_k follows from φ_i and φ_j by MP. This means that $\varphi_1\ldots\varphi_\ell$ also satisfies the definition of a deduction from Σ. ■

3.7. *[Proposition 3.7]* If $\Gamma \vdash \Delta$ and $\Delta \vdash \sigma$, then $\Gamma \vdash \sigma$. *[3]*

Solution. This is as much an exercise in trying to find suitable notation as anything else. The basic idea is to take a deduction of σ from Δ and insert the deduction from Γ of each premiss in Δ in place of that premiss in the deduction of σ.

Let $\eta_1\eta_2\ldots\eta_n$ be a deduction of σ from Δ. For each $k \leq n$, let $\eta_1^k\eta_2^k\ldots\eta_{m_k}^k$ be

- just η_k if η_k is an axiom or follows from preceding η_i and η_j by Modus Ponens (so $n_k = 1$ and η_{m_k} is η_k in either case), and
- a deduction of η_k from Γ (so η_{m_k} is η_k) if $\eta_k \in \Delta$.

Each formula in the sequence $\eta_1^1\eta_2^1\ldots\eta_{m_1}^1\eta_1^2\eta_2^2\ldots\eta_{m_2}^2\ldots\eta_1^n\eta_2^n\ldots\eta_{m_n}^n$ is then either an axiom, follows from preceding formulas in the sequence by Modus Ponens, or is a premiss from Γ. Since the last formula in the sequence is $\eta_{m_n}^n$, *a.k.a.* η_n, *a.k.a.* σ, it is a deduction of σ from Γ, so $\Gamma \vdash \sigma$, as desired. ■

3.8. *[Theorem 3.8 – Deduction Theorem]* If Σ is any set of formulas and α and β are any formulas, then $\Sigma \vdash \alpha \to \beta$ if and only if $\Sigma \cup \{\alpha\} \vdash \beta$. *[5]*

Solution. ($\Longrightarrow$) Suppose $\Sigma \vdash \alpha \to \beta$, say via a deduction $\varphi_1\varphi_2\ldots\varphi_n$ (so φ_n is $\alpha \to \beta$). Then $\varphi_1\varphi_2\ldots\varphi_n\alpha\beta$ is a deduction of β from $\Sigma \cup \{\alpha\}$ because each formula in the sequence is either a premiss in $\Sigma \cup \{\alpha\}$, an axiom, or follows from preceding formulas in the sequence by Modus Ponens. Note that β, in particular, follows from φ_n, *i.e.* $\alpha \to \beta$, and α (which is in $\Sigma \cup \{\alpha\}$) by Modus Ponens.

($\Longleftarrow$) Suppose $\Sigma \cup \{\alpha\} \vdash \beta$ for some formula β, say via a shortest deduction $\varphi_1\varphi_2 \ldots \varphi_n$ (so φ_n is β). We will use induction on $n \geq 1$ to show that $\Sigma \vdash \alpha \to \beta$.

Base Step. ($n = 1$) In this case β is either an axiom or a premiss (*i.e.* $\beta \in \Sigma \cup \{\alpha\}$). If β is an axiom or $\beta \in \Sigma$, then

1.	$\beta \to (\alpha \to \beta)$	(A1)
2.	β	Axiom or $\beta \in \Sigma$
3.	$\alpha \to \beta$	1,2 MP

is a deduction of $\alpha \to \beta$ from Σ. If β is α, the other way β could be a premiss from $\Sigma \cup \{\alpha\}$, then $\vdash \alpha \to \alpha$ by Example 3.1 in the textbook, so $\Sigma \vdash \alpha \to \beta$ because $\emptyset \subseteq \Sigma$ and the fact that β is α.

In each case, we can conclude that $\Sigma \vdash \alpha \to \beta$ in the Base Step.

Induction Hypothesis. ($n \leq k$) Assume that for all formulas β with $\Sigma \cup \{\alpha\} \vdash \beta$ via a shortest possible deduction $\varphi_1\varphi_2 \ldots \varphi_n$ for some n with $1 \leq n \leq k$, we have $\Sigma \vdash \alpha \to \beta$.

Induction Step. ($n = k + 1$) Suppose that $\Sigma \cup \{\alpha\} \vdash \beta$ for some formula β, via a shortest deduction $\varphi_1\varphi_2 \ldots \varphi_k\varphi_{k+1}$ (so φ_{k+1} is β). Since this is the shortest possible deduction and $k \geq 1$, φ_{k+1}, otherwise known as β, must have been obtained from some φ_i and φ_j with $i, j \leq k$ by Modus Ponens. Without loss of generality, we may suppose that φ_i is $\varphi_j \to \beta$. By Proposition 3.4, $\varphi_1\varphi_2 \ldots \varphi_i$ and $\varphi_1\varphi_2 \ldots \varphi_j$ are also deductions from $\Sigma \cup \{\alpha\}$, so $\Sigma \vdash \alpha \to \varphi_i$, *i.e.* $\Sigma \vdash \alpha \to (\varphi_j \to \beta)$, and $\Sigma \vdash \alpha \to \varphi_j$ by the Induction Hypothesis. Suppose $\eta_1\eta_2 \ldots \eta_\ell$ and $\zeta_1\zeta_2 \ldots \zeta_m$ are deductions of $\alpha \to (\varphi_j \to \beta)$ and $\alpha \to \varphi_j$, respectively, from Σ. Then

$$
\begin{array}{rll}
1. & \eta_1 & \\
 & \vdots & \\
\ell. & \eta_\ell & \quad \textit{i.e. } \alpha \to (\varphi_j \to \beta) \\
\ell + 1. & \zeta_1 & \\
 & \vdots & \\
\ell + m. & \zeta_m & \quad \textit{i.e. } \alpha \to \varphi_j \\
\ell + m + 1. & (\alpha \to (\varphi_j \to \beta)) \to ((\alpha \to \varphi_j) \to (\alpha \to \beta)) & \text{(A2)} \\
\ell + m + 2. & (\alpha \to \varphi_j) \to (\alpha \to \beta) & \ell,\ \ell + m + 1 \text{ MP} \\
\ell + m + 3. & \alpha \to \beta & \ell + m,\ \ell + m + 2 \text{ MP}
\end{array}
$$

is a deduction of $\alpha \to \beta$ from Σ, so $\Sigma \vdash \alpha \to \beta$, as desired.

Thus, by induction, if $\Sigma \cup \{\alpha\} \vdash \beta$, then $\Sigma \vdash \alpha \to \beta$. ■

3.9(3). *[Proposition 3.9(3)]* Appealing to previous deductions and the Deduction Theorem if you wish, show that $\vdash (\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$. *[2]*

NOTE. You may assume any and all the examples, problems, and results of Chapter 3, up to **3.9**(2) inclusive, when doing **3.9**(3).

SOLUTION. By the Deduction Theorem, applied twice:

$$\begin{aligned} &\vdash (\neg\beta \to \neg\alpha) \to (\alpha \to \beta) \\ \iff &\{\neg\beta \to \neg\alpha\} \vdash \alpha \to \beta \\ \iff &\{\neg\beta \to \neg\alpha,\ \alpha\} \vdash \beta \end{aligned}$$

It therefore suffices to show that $\{\neg\beta \to \neg\alpha,\ \alpha\} \vdash \beta$, which do via the following deduction:

1.	$(\neg\beta \to \neg\alpha) \to ((\neg\beta \to \alpha) \to \beta)$	(A3)
2.	$\neg\beta \to \neg\alpha$	Premiss
3.	$(\neg\beta \to \alpha) \to \beta$	1, 2 MP
4.	$\alpha \to (\neg\beta \to \alpha)$	(A1)
5.	α	Premiss
6.	$\neg\beta \to \alpha$	4, 5 MP
7.	β	3, 6 MP

That's all folks!. ■

[Total = 15]