Mathematics-Computer Science 4215H – Mathematical Logic
Trent University, Winter 2021

Solutions to Assignment #2
Due on Friday, 29 January.

Do all of the following problems, which are straight out of Chapter 1 of the textbook[0] (which explains the numbering), reproduced here for your convenience.

2.1. *[Problem 2.1]* Suppose v is a truth assignment such that $v(A_0) = v(A_2) = T$ and $v(A_1) = v(A_3) = F$. Find $v(\alpha)$ if α is:

(1) $\neg A_2 \to \neg A_3$
(3) $\neg(\neg A_0 \to A_1)$
(5) $A_0 \wedge A_1$ *[1.5 = 3×0.5 each]*

Solution. (1) From the definition of how truth assignments handle the connectives, $v(\neg A_2) = F$ since $v(A_2) = T$ and $v(\neg A_3) = T$ since $v(A_3) = F$, and then it follows that $v(\neg A_2 \to \neg A_3) = T$.

(2) $v(\neg A_0) = F$ since $v(A_0) = T$, and $v(A_1) = F$, so $v(\neg A_0 \to A_1) = T$, and then it follows that $v(\neg(\neg A_0 \to A_1)) = F$.

(3) $A_0 \wedge A_1$ is short for $\neg(A_0 \to (\neg A_1))$. $v(A_0) = T$ and $v(A_1) = F$, so $v(\neg A_1) = T$, and thus $v(A_0 \to (\neg A_1)) = T$, and hence $v(A_0 \wedge A_1) = v(\neg(A_0 \to (\neg A_1))) = F$. ■

2.2. *[Proposition 2.2]* Suppose δ is any formula and u and v are truth assignments such that $u(A_n) = v(A_n)$ for all atomic formulas A_n which occur in δ. Show that $u(\delta) = v(\delta)$. *[4.5]*

Solution. We will proceed by induction on the number of connectives, call it c, in the formula δ.

Base Step: $(c = 0)$ If δ has no connectives, it must be an atomic formula A_n for some n. The fact that $u(A_n) = v(A_n)$ for all atomic formulas A_n which occur in δ then means that $u(\delta) = u(A_n) = v(A_n) = v(\delta)$.

Induction Hypothesis: $(0 \leq c \leq k)$ For any formula δ with $\leq k$ connectives for which $u(A_n) = v(A_n)$ for all atomic formulas A_n which occur in δ, we have $u(\delta) = v(\delta)$.

Induction Step: $(c = k+1)$ Suppose δ is a formula with $k+1$ connectives and $u(A_n) = v(A_n)$ for all atomic formulas A_n which occur in δ. Since δ has $k+1 \geq 1$ connectives, it must be either (i) $(\neg\alpha$ for a subformula α with k connectives, or (ii) $(\beta \to \gamma)$ for subformulas β and γ which have k connectives between them, and hence $\leq k$ connectives each. In both cases, since $u(A_n) = v(A_n)$ for all atomic formulas A_n which occur in δ, the truth assignments must also agree on all the atomic formulas which occur in the subformulas. In case (i) it follows that $u(\alpha) = v(\alpha)$ by the induction hypothesis, so $u(\delta) = u(\neg\alpha) = v(\neg\alpha) = v(\delta)$ by the definition of how truth assignments ineract with $\neg$; in case (ii) it follows that $u(\beta) = v(\beta)$ and $u(\gamma) = v(\gamma)$ by the induction hypothesis, so $u(\delta) = u(\beta \to \gamma) = v(\beta \to \gamma) = v(\delta)$ by the definition of how truth assignments ineract with $\to$.

[0] *A Problem Course in Mathematical Logic*, Version 1.6.

Thus, by induction, $u(\delta) = v(\delta)$ if $u(A_n) = v(A_n)$ for all atomic formulas A_n which occur in δ. ■

2.3. *[Corollary 2.3]* Suppose u and v are truth assignments such that $u(A_n) = v(A_n)$ for every atomic formula A_n. Show that $u = v$, *i.e.* $u(\varphi) = v(\varphi)$ for every formula φ. *[1]*

SOLUTION. If the truth assignments u and v agree on all atomic formulas, they must agree on all atomic formulas that occur in any given formula δ, but then $u(\delta) = v(\delta)$ by Proposition 2.2 above. Since u and v agree on all formulas, $u = v$. ■

2.7. *[Proposition 2.7]* Show that if Γ and Σ are sets of formulas such that $\Gamma \subseteq \Sigma$, then $\Sigma \models \Gamma$. *[1]*

SOLUTION. Suppose a truth assignment u makes every formula in Σ true. Then it must also make every formula in Γ true, since $\Gamma \subseteq \Sigma$ means that every formula in Γ is also a formula in Σ. ■

2.8. *[Problem 2.8]* How can one check whether or not $\Sigma \models \varphi$ for a formula φ and a finite set of formulas Σ? *[3]*

SOLUTION. Note that because formulas are finite and $\Sigma \cup \{\varphi\}$ is a finite set of formulas, only finitely many atomic formulas can occur in the formulas of $\Sigma \cup \{\varphi\}$. A simple and effective, if brutally inefficient, method to solve the problem is to grind out a truth table for all of the formulas of $\Sigma \cup \{\varphi\}$ and use it to evaluate the truth or falsity of all the formulas of Σ and of φ on all possible ways truth assignments could assign truth values to these finitely many atomic formulas. (This is the point of Proposition 2.2 above.) If every row of the truth table in which every formula of Σ is true also has φ true, then $\Sigma \models \varphi$, and otherwise $\Sigma \not\models \varphi$.

This method is brutally inefficient because the size of the truth table grows exponentially with the number of atomic formulas that occur in the formulas of $\Sigma \cup \{\varphi\}$. There are various ways to make the method more efficient, but is not clear if the problem can be done in polynomial time, since checking satisfiability is an NP-complete problem. ■

2.10. *[Proposition 2.10]* Show that a set of formulas Σ is satisfiable if and only if there is no contradiction χ such that $\Sigma \models \chi$. *[4]*

SOLUTION. ($\Longrightarrow$) Suppose the truth assignment u satisfies the set of formulas Σ, *i.e.* $u(\sigma) = T$ for every $\sigma \in \Sigma$. By the definition of $\models$, $\Sigma \models \chi$ would then require that $u(\chi) = T$, but for any contradiction χ, we must have $u(\chi) = F$ by the definition of contradiction. Thus there can be no contradiction χ with $\Sigma \models \chi$ if Σ is satisfiable.

($\Longleftarrow$) We will actually prove the contrapositive, *i.e.* that if there is a contradiction χ such that $\Sigma \models \chi$, then Σ is not satisfiable. Assume, then, that χ is a contradiction such that $\Sigma \models \chi$. By the definition of $\models$, $\Sigma \models \chi$ means that we would have $u(\chi) = T$ for any truth assignment u that satisfies Σ. Since, by the definition of contradiction, $u(\chi) = F$ for every truth assignment u, there cannot be any truth assignment that satisfies Σ. ■

[Total = 15]